School of Engineering
Brown University

EN1750: Advanced Mechanics of Solids

Homework 4: Internal forces and equations of motion.

Due Friday Oct 5, 2018

1. For the Cauchy stress tensor with components

$$\boldsymbol{\sigma} = \begin{bmatrix} 50 & 250 & 0 \\ 250 & 130 & -10 \\ 0 & -10 & 320 \end{bmatrix}$$

compute

1. The traction vector acting on an internal material plane with normal $\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}\mathbf{e}_2$
2. The principal stresses
3. The hydrostatic stress
4. The deviatoric stress tensor
5. The Von-Mises equivalent stress (find the answer using the answers to both (2) and (4))

You can use MATLAB for this – if you do, just paste a screenshot of your matlab script and its solutions into the pdf or word document you use to submit your solutions.

This is just a matter of using the definitions. The numbers are best crunched using MATLAB. A solution is shown below

```
syms n sigma
sigma = [50, 250, 0;...
         250, 130, -10;...
         0, -10, 320]
n = [1,-1,0]/sym(2)^(1/2)
traction = n*sigma
ps = eig(sigma)
hydrostatic = trace(sigma)/3
sdev = sigma - eye(3)*hydrostatic
smises = sqrt(((ps(1)-ps(2))^2 + (ps(1)-ps(3))^2 + (ps(2)-ps(3))^2)/2)
smises = sqrt(3*sum(sum(sdev.*sdev))/2)
```

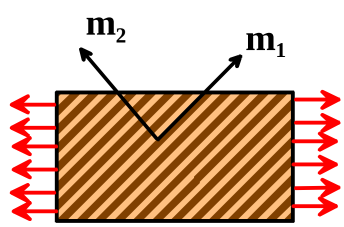
traction = $\left(-100\sqrt{2}\quad 60\sqrt{2}\quad 5\sqrt{2}\right)$

```
ps = 3×1
   -163.2669
    317.8057
    345.4612

hydrostatic = 166.6667
sdev = 3×3
   -116.6667   250.0000          0
    250.0000   -36.6667   -10.0000
           0   -10.0000   153.3333

smises = 495.4796
```

smises = 495.4796

[5 POINTS]

2. The ‘Tsai-Hill’ criterion is used to predict the critical loads that cause failure in brittle laminated fiber-reinforced composites and wood. A specimen of laminated composite subjected to in-plane loading is sketched in the figure. The Tsai-Hill criterion assumes that a plane stress state exists in the solid. Let $\sigma_{11}, \sigma_{22}, \sigma_{12}$ denote the nonzero components of stress, with basis vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ oriented parallel and perpendicular to the fibers in the sheet, as shown. The Tsai-Hill failure criterion is



$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}^2} + \frac{\sigma_{12}^2}{\sigma_{SS}^2} = 1$$

at failure, where σ_{TS1}, σ_{TS2} and σ_{SS} are material properties. (The material fails when the quantity on the left hand side of the equation is equal to 1). This report gives the following data for the properties of a graphite-epoxy CFRP

Property	Value (MPa)
σ_{TS1}	2206
σ_{TS2}	56.5
σ_{SS}	110.3

Suppose that a specimen is loaded in uniaxial tension with the tensile axis at 45 degrees to the fiber direction, as shown in the figure. Calculate the maximum stress that the material can withstand (you will need to use the basis change formulas for a tensor).

The general formula relating tensor components in two bases is

$$\begin{bmatrix} S_{11}^{(\mathbf{m})} & S_{12}^{(\mathbf{m})} & S_{13}^{(\mathbf{m})} \\ S_{21}^{(\mathbf{m})} & S_{22}^{(\mathbf{m})} & S_{23}^{(\mathbf{m})} \\ S_{31}^{(\mathbf{m})} & S_{32}^{(\mathbf{m})} & S_{33}^{(\mathbf{m})} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1\cdot\mathbf{e}_1 & \mathbf{m}_1\cdot\mathbf{e}_2 & \mathbf{m}_1\cdot\mathbf{e}_3 \\ \mathbf{m}_2\cdot\mathbf{e}_1 & \mathbf{m}_2\cdot\mathbf{e}_2 & \mathbf{m}_2\cdot\mathbf{e}_3 \\ \mathbf{m}_3\cdot\mathbf{e}_1 & \mathbf{m}_3\cdot\mathbf{e}_2 & \mathbf{m}_3\cdot\mathbf{e}_3 \end{bmatrix} \begin{bmatrix} S_{11}^{(\mathbf{e})} & S_{12}^{(\mathbf{e})} & S_{13}^{(\mathbf{e})} \\ S_{21}^{(\mathbf{e})} & S_{22}^{(\mathbf{e})} & S_{23}^{(\mathbf{e})} \\ S_{31}^{(\mathbf{e})} & S_{32}^{(\mathbf{e})} & S_{33}^{(\mathbf{e})} \end{bmatrix} \begin{bmatrix} \mathbf{m}_1\cdot\mathbf{e}_1 & \mathbf{m}_2\cdot\mathbf{e}_1 & \mathbf{m}_3\cdot\mathbf{e}_1 \\ \mathbf{m}_1\cdot\mathbf{e}_2 & \mathbf{m}_2\cdot\mathbf{e}_2 & \mathbf{m}_3\cdot\mathbf{e}_2 \\ \mathbf{m}_1\cdot\mathbf{e}_3 & \mathbf{m}_2\cdot\mathbf{e}_3 & \mathbf{m}_3\cdot\mathbf{e}_3 \end{bmatrix}$$

Here

$$\frac{\sigma}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\sigma_{11} = \sigma_{22} = \sigma / 2 \qquad \sigma_{12} = -\sigma / 2$

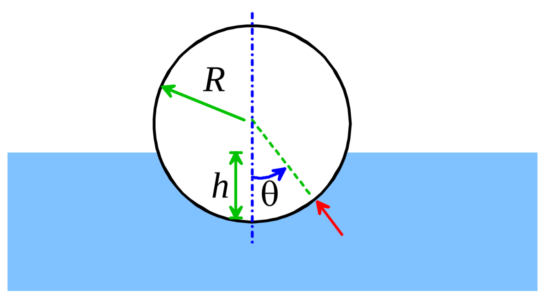
$$\left(\frac{\sigma/2}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma/2}{\sigma_{TS2}}\right)^2 - \frac{\sigma^2/4}{\sigma_{TS1}^2} + \frac{\sigma^2/4}{\sigma_{SS}^2} = 1$$

$$\Rightarrow \frac{\sigma^2}{4}\left(\frac{1}{\sigma_{TS2}^2} + \frac{1}{\sigma_{SS}^2}\right) = 1$$

$$\Rightarrow \sigma = 100.6MPa$$

[5 POINTS]

3. A cylinder of radius *R* is partially immersed in a static fluid.



(3.1) Recall that the pressure at a depth *d* in a fluid has magnitude rgd. Write down an expression for the horizontal and vertical components of traction acting on the surface of the cylinder in terms of q.

The pressure is $p = \rho g\left[h - R\left(1 - \cos\theta\right)\right]$

The traction acts radially so

$$t_1 = -p\sin\theta = -\rho g\left(h - R(1-\cos\theta)\right)\sin\theta$$

$$t_2 = p\cos\theta = \rho g\left(h - R(1-\cos\theta)\right)\cos\theta$$

[2 POINTS]

(3.2) Hence compute the resultant force (per unit out of plane distance) exerted by the fluid on the cylinder, in terms of ρ, g, h, R.

The total force will be

$$F_1 = \int_{-\phi}^{\phi} t_1 R d\theta \qquad F_2 = \int_{-\phi}^{\phi} t_2 R d\theta \qquad \phi = \cos^{-1}(1 - h/R)$$

Evaluating the integrals gives

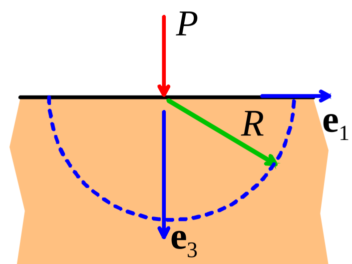
$F_1 = 0$ (obvious)

$$F_2 = \rho g \left[R^2 \cos^{-1}\left(1 - \frac{h}{R}\right) - (R-h)\sqrt{R^2 - (R-h)^2} \right]$$

You can check the answer by setting h=R – in this case the volume of fluid displaced is obviously $\pi R^2 / 2$

[3 POINTS]

4. The stresses in a large incompressible elastic solid with a flat surface, which has a point force magnitude P acting vertically downwards at the origin is given by



$$\sigma_{ij} = \frac{-3P x_i x_j x_3}{2\pi R^5}$$

4.1 Show that the stress state satisfies the equilibrium equation

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$

Recall that

$$\frac{\partial R}{\partial x_i} = \frac{x_i}{R} \qquad \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

Therefore

$$\frac{\sigma_{ij}}{\partial x_i} = \frac{-3P}{2\pi R^5}\left(\delta_{ii} x_j x_3 + x_i \delta_{ij} x_3 + x_i x_j \delta_{i3} - \frac{x_i x_j x_3 x_i}{R^2} \right) = 0$$

Where we noted that $\delta_{ii} = 3 \quad x_i x_i = R^2$

[3 POINTS]

4.2 Find a formula for the traction acting on a hemispherical surface with radius R inside the solid (note that x_i / R is a unit vector normal to the surface).

$$t_j = n_i \sigma_{ij} = \frac{x_i}{R} \frac{-3P x_i x_j x_3}{2\pi R^5} = \frac{-3P x_j x_3}{2\pi R^4}$$

[2 POINTS]

5. Importing CAD data into ABAQUS. Work through the tutorial on this topic. Then, use Solidworks to create a simple part, import it into ABAQUS (try both a step or iges import and a .inp file import. Its best to avoid the STL file import), and run a simulation. Keep everything simple, or you will run into problems. Try to make your part differ from those submitted by your classmates!

Please upload your solidworks model and cae file to canvas as a solution to this problem.

[20 POINTS]