**1.** A solid cylindrical bar with radius $a$ and length $L$ is subjected to a uniform pressure $p$ on its ends. The bar is made from a linear elastic solid with Young's modulus $E$ and Poisson's ratio $v$.



   1.1 Write down an equation for the stress in the bar. Show that the stress satisfies the equation of static equilibrium, and the boundary conditions $\sigma_{ij} n_i = t_j$ on all its surfaces. Express your answer as components in a Cartesian basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1$ parallel to the axis of the cylinder.
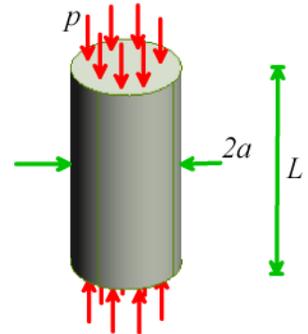
This is a trivial problem – the bar is in a state of uniaxial tension $\sigma_{11} = -p$ with all other components zero.

The side face has a normal vector $\mathbf{n} = (x_2 / a)\mathbf{e}_2 + (x_3 / a)\mathbf{e}_3$ and so $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0}$ , while on the end faces the normal vector is parallel to $\mathbf{e}_1$ so $\mathbf{n} \cdot \boldsymbol{\sigma} = -p$

**[1 POINT]**

   1.2 Find the strain in the bar (neglect temperature changes)

The elastic stress-strain relations give $\varepsilon_{11} = -p / E$     $\varepsilon_{22} = \varepsilon_{33} = v / E$ ; all other components zero.

**[1 POINTS]**

   1.3 Find the displacement field in the bar

$$\frac{\partial u_1}{\partial x_1} = \varepsilon_{11} \Rightarrow u_1 = -px_1 / E$$

$$\frac{\partial u_2}{\partial x_2} = \varepsilon_{22} \Rightarrow u_2 = v\,px_2 / E$$

$$\frac{\partial u_3}{\partial x_3} = \varepsilon_{33} \Rightarrow u_3 = v\,px_3 / E$$

**[1 POINT]**

   1.4 Calculate a formula for the change in length of the bar

$$\Delta L = u_1(L) = -pL / E$$

**[1 POINT]**

1.5 Find a formula for the stiffness of the bar (stiffness = force/extension)

The force is $-p\pi a^2 \mathbf{e}_1$ ; therefore $k = L / (E\pi a^2)$

**[1 POINT]**

1.6 Find the change in volume of the bar

This can be done in many different ways – the volumetric strain is $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = -(1-2\nu)p / E$ ; this is the fractional change in volume. So the change in volume is $\Delta V = -\pi a^2 L(1-2\nu)p / E$

Alternatively the original volume is $\pi a^2 L$ ,

the deformed volume is $\pi\left[a(1+\nu p / E)\right]^2 L\left[1 - p / E\right] \approx \pi a^2 L(1 - p(1-2\nu) / E)$ , and we get the same result.

**[1 POINT]**

1.7 Calculate the total strain energy in the bar.

The strain energy density is $\dfrac{1}{2}\sigma_{ij}\varepsilon_{ij} = p^2 / 2E$ so the total strain energy is

$$W = \pi a^2 L p^2 / 2E$$

**[1 POINT]**

2. A specimen of material is placed inside a rigid box that prevents the material from stretching in any direction. This means that the strains in the specimen are zero. The specimen is then heated to increase its temperature by $\Delta T$. Find a formula for the stress in the specimen. Find a formula for the strain energy density. How much strain energy would be stored in a $1cm^3$ sample of steel if its temperature were increased by 100C? Compare the strain energy with the heat required to change the temperature by 100C – the specific heat capacity of steel is about 470 J/(kg-C)

The elastic stress-strain equations are

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2v)}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2v} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

The strains are all zero, so

$$
\sigma_{11} = \sigma_{22} = \sigma_{33} = -\frac{E\alpha\Delta T}{1-2v}
$$

The strain energy density is $\frac{1}{2}\sigma_{ij}\varepsilon_{ij}^e = \frac{3}{2}\frac{E}{1-2v}(\alpha\Delta T)^2$

Steel has a Youngs modulus of about 210GPa, thermal expansion coefficient about $15 \times 10^{-6} \mathrm{K}^{-1}$ and Poissons ratio about 0.3. So this gives 1.778 MJ/m$^3$. So a 1cm$^3$ sample has 1.77Joules of strain energy....

The mass of a 1 cm$^3$ sample is 0.0081kg; the energy required to heat this by 100C is 378J.

**[4 POINTS]**

**3.** Amber Kinetics offers a flywheel energy storage system. The goal of this problem is to work through some rough design calculations for their device.

Idealize the flywheel as a hollow cylinder with mass density $\rho$ that spins at constant angular speed $\omega$. The rotational accelerations will induce a stress field in the cylinder. Work through the following steps to calculate this stress field

3.1 Assume that the displacement field in the cylinder has the form $\mathbf{u} = u(r)\mathbf{e}_r + \varepsilon_{zz}z\mathbf{e}_z$ ($u(r)$ is a function to be determined, and $\varepsilon_{zz}$ is a constant axial strain to be found). Find a formula for the infinitesimal strain in the cylinder in terms of $u$ and its derivatives and $\varepsilon_{zz}$. Use cylindrical polar coordinates.
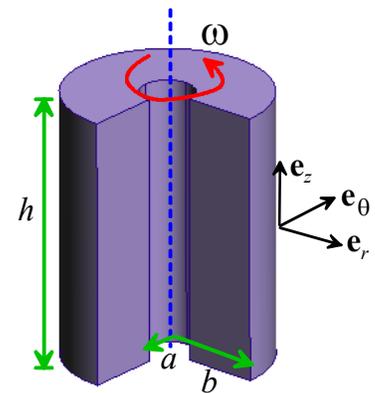
We have that $\boldsymbol{\varepsilon} = \frac{1}{2}\{\nabla\mathbf{u} + (\nabla\mathbf{u})^T\}$. We can use the formula for cylindrical-polar gradients

$$\nabla(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z) \equiv \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r} & \dfrac{\partial v_r}{\partial z} \\[2mm] \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r} & \dfrac{\partial v_\theta}{\partial z} \\[2mm] \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} \Rightarrow \boldsymbol{\varepsilon} = \begin{bmatrix} du/dr & 0 & 0 \\ 0 & u/r & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$

**[1 POINT]**

3.2 Hence, find a formula for the stress in the cylinder, in terms of $u(r)$ and its derivatives and $\varepsilon_{zz}$

The linear elastic stress-strain relations give $\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix} = \dfrac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v \\ v & 1-v & v \\ v & v & 1-v \end{bmatrix} \begin{bmatrix} \dfrac{du}{dr} \\[2mm] \dfrac{u}{r} \\[2mm] \varepsilon_{zz} \end{bmatrix}$

**[1 POINT]**

3.3 Assume that the cylinder is rotating at constant angular speed (with no vibrations or any other kinds of motion), and assume the displacement is small, so the acceleration vector is (approximately) $\mathbf{a} = -r\omega^2\mathbf{e}_r$ (from the circular motion formula). Hence (using the linear momentum balance equation in polar coordinates) show that the displacement u® must satisfy

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru)\right\} = -\frac{(1+v)(1-2v)}{E(1-v)}\rho\omega^2 r$$

(It is possible to do the algebra by hand but it's a bit nasty – Matlab or mathematica will do it for you if you like).

We need to substitute (3.1) into (3.2) and then substitute the stresses into the EOM in cylindrical polar coordinates. From class, the linear momentum balance equation in cylindrical polar coords is

$$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \equiv \begin{bmatrix} \dfrac{\partial\sigma_{rr}}{\partial r} + \dfrac{\sigma_{rr}}{r} + \dfrac{1}{r}\dfrac{\partial\sigma_{\theta r}}{\partial\theta} + \dfrac{\partial\sigma_{zR}}{\partial z} - \dfrac{\sigma_{\theta\theta}}{r} \\[2mm] \dfrac{1}{r}\dfrac{\partial\sigma_{\theta\theta}}{\partial\theta} + \dfrac{\partial\sigma_{r\theta}}{\partial r} + \dfrac{\sigma_{r\theta}}{r} + \dfrac{\sigma_{\theta r}}{r} + \dfrac{\partial\sigma_{z\theta}}{\partial z} \\[2mm] \dfrac{\partial\sigma_{zz}}{\partial z} + \dfrac{\partial\sigma_{rz}}{\partial r} + \dfrac{\sigma_{rz}}{r} + \dfrac{1}{r}\dfrac{\partial\sigma_{\theta z}}{\partial\theta} \end{bmatrix} + \begin{bmatrix} \rho b_r \\[2mm] \rho b_\theta \\[2mm] \rho bz \end{bmatrix} = \begin{bmatrix} \rho\dfrac{dv_r}{dt} \\[2mm] \rho\dfrac{dv_\theta}{dt} \\[2mm] \rho\dfrac{dv_z}{dt} \end{bmatrix}$$

Only the first equation is nonzero

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = -\rho\omega^2 r$$

Here's a MATLAB solution, which can be rearranged by hand into the final form

```
clear all
syms rho omega EE nu C1 C2 ezz srr sqq real
syms u(r)
C1 = EE/(1+nu)/(1-2*nu);
srr = C1*((1-nu)*diff(u(r),r) + nu*u(r)/r + nu*ezz);
sqq = C1*(nu*diff(u(r),r) + (1-nu)*u(r)/r + nu*ezz);
simplify(diff(srr,r) + (srr-sqq)/r )== -rho*omega^2*r
```

ans =

$$\frac{EE\,(v-1)\left(r^2\dfrac{\partial^2}{\partial r^2}u(r) - u(r) + r\dfrac{\partial}{\partial r}u(r)\right)}{r^2(2v^2+v-1)} = -\omega^2 r\rho$$

**[3 POINTS]**

3.4 Finally, for boundary conditions assume that
- The radial displacement at $r=a$ is zero,
- The radial stress at $r=b$ is zero.
- The axial force $F_z = \int_a^b 2\pi\sigma_{zz} r \, dr$ is zero.

Use your favorite symbolic manipulation program to solve the differential equation in 3.3 for the displacement field with these boundary conditions and hence find formulas for the stresses (you can download and modify the solution for the spherically symmetric problem solved class if you want to use Matlab). **Simplify your final expression by substituting** $a=0$ (to produce a solid cylinder)

Solving these equations (even in a symbolic manipulation program) is a bit of a chore. Your calculation could follow these steps:

(1) Integrate 3.3 with respect to r twice to find a formula for u(r) – the answer will contain two arbitrary constants

(2) Substitute the solution for u from (1) into the formula for $\sigma_{zz}$ in terms of u(r) and $\varepsilon_{zz}$

(3) Integrate (2) to find a formula for the axial force (still in terms of u(r)).

(4) Solve the result of (3) and the condition $F_z = 0$ for the unknown axial strain $\varepsilon_{zz}$.

(5) Substitute the result of (4) into the result of (1) to find a formula for $\sigma_{rr}$ in terms of u(r)

(6) Use the results of (5) and (1) and the conditions $u(a) = 0$, $\sigma_{rr}(b) = 0$ to get two equations for the two unknown constants of integration

(7) Solve the equations for the constants, and hence find u(r)

(8) Substitute the results of (7) , as well as $a=0$ into the formulas for stress and simplify the results.


Hence, plot a graph showing

(a) The normalized Von-Mises stress $\dfrac{\sigma_e}{\rho\omega^2 b^2} = \dfrac{\sqrt{\dfrac{1}{2}\left[(\sigma_{rr} - \sigma_{zz})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{rr} - \sigma_{\theta\theta})^2\right]}}{\rho\omega^2 b^2}$

(b) The normalized stresses $\dfrac{\sigma_{rr}}{\rho\omega^2 b^2}, \dfrac{\sigma_{\theta\theta}}{\rho\omega^2 b^2}, \dfrac{\sigma_{zz}}{\rho\omega^2 b^2}$

as functions of normalized radial position $r/b$ in the disk, with Poisson's ratio $\nu = 0.3$ . Hand in your solutions for the displacement field and stress fields, and the graph

u(r) =

$$-\frac{\omega^2 r \rho \left(2 v^2 + v - 1\right) \left(2 b^2 v^2 - 7 b^2 v + 3 b^2 + 3 v r^2 - r^2\right)}{8 \, EE \, (3 v - 1)(v - 1)}$$

sqq =

$$\frac{\omega^2 \rho \left(2 b^2 v + 2 v r^2 - 3 b^2 + r^2\right)}{8 \, (v - 1)}$$

srr =

$$\frac{\omega^2 \rho \, (2 v - 3) \left(b^2 - r^2\right)}{8 \, (v - 1)}$$

szz =

$$-\frac{v \omega^2 \rho \left(b^2 - 2 r^2\right)}{4 \, (v - 1)}$$
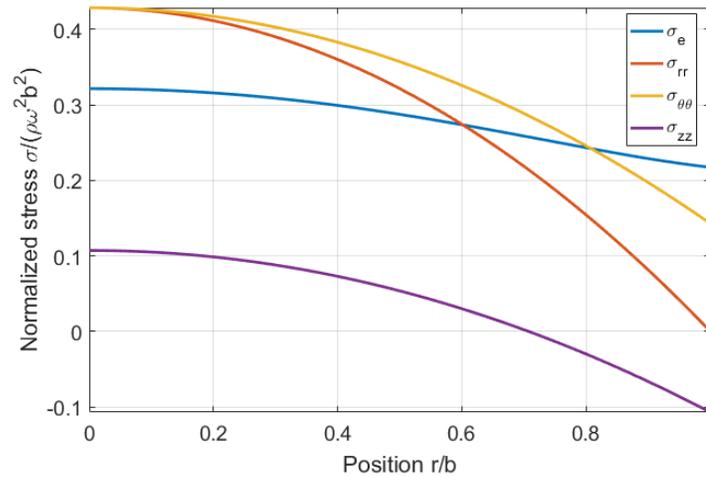


**[5 POINTS]**

3.2 Suppose that the material in the disk is made from a metal, so that the device must be designed and operated to ensure that the Von-Mises stress $\sigma_e$ stays below a critical value $Y$ (the yield stress of the metal). Show that the maximum kinetic energy that can be stored in the disk is

$$U_{max} = \beta Y V$$

where $V$ is the volume of the cylinder and $\beta$ is a constant that depends only on Poisson's ratio. Find the formula for $\beta$

Just in case you don't remember this from ENGN40 - the kinetic energy in the disk is $\frac{1}{2}I_{zz}\omega^2$ where

$I_{zz} = \frac{1}{2}mb^2$ is its mass moment of inertia about the axis of rotation.

Re-writing the KE in terms of density gives

$$U = \frac{1}{4}\rho\omega^2 b^2 V$$

The preceding problem shows that the maximum Mises stress occurs at $r=0$ (if you are super-cautious you can check that this is still true for all physically admissible values of Poissons ratio), where

$$\sigma_e = \frac{\rho\omega^2 b^2}{8}\frac{(3-4v)}{(1-v)}$$

To avoid yield we must design and operate the device so that

$$\frac{\rho\omega^2 b^2}{8}\frac{(3-4v)}{(1-v)} < Y \Rightarrow \omega^2 b^2 < 8Y\frac{(1-v)}{(3-4v)}$$

Hence

$$U < 2Y\frac{(1-v)}{(3-4v)}V$$

It follows that $\beta = 2\frac{(1-v)}{(3-4v)}$

**[5 POINTS]**

3.3 Amber Kinetics say their disk is made from steel, and give the following specifications for their device:
- Weight: 10000lb
- Dimensions Width 54 inches and height 52 inches.
- Energy stored: 32 kWhr.
Steel has a density of 8050kg/m$^3$

Estimate the yield stress of their steel.

Converting 10000lb to kg gives m=4536kg; steel density is 8050kg/m$^3$
The energy is 115.2 MJ

```
beta = 2*(1-0.3)/(3-4*0.3);
V = 4535.9/8050;
1.152e08/V/beta
```

This gives Y=263MPa – this seems like rather a conservative value but the designers may have kept the stresses low to ensure a long fatigue life.

**[3 POINTS]**

**4.** Show that the function

$$\phi = \frac{\sigma_0}{4}\left(-2a^2\log(r)+r^2\right)+\frac{\sigma_0}{4}\left(+2a^2-r^2-\frac{a^4}{r^2}\right)\cos 2\theta$$

is a valid Airy function (i.e. show that it satisfies the biharmonic equation). You can use Matlab or Mathematica if you want. If you do this please upload a copy of your MATLAB or mathematica script to Canvas as part of your HW submission (there is a separate link for this upload)

Here's a MATLAB demo
```
clear all
syms phi s0 a r theta srr sqq srq real
phi = s0/4*(-2*a^2*log(r)+r^2) + s0/4*(2*a^2-r^2-a^4/r^2)*cos(2*theta);
diff1 = diff(phi,r,2) + diff(phi,r)/r + diff(phi,theta,2)/r^2;
diff2 = simplify(diff(diff1,r,2) + diff(diff1,r)/r + diff(diff1,theta,2)/r^2)
```

This shows that diff2=0, which means the equation is satisfied.

**[2 POINTS]**

Next. show that the function generates a stress state that represents the solution to a large plate containing a circular hole with radius $a$ at the origin, which is loaded by a tensile stress $\sigma_0$ acting parallel to the $\mathbf{e}_1$ direction. To do this,

4.1 Show that the surface of the hole is traction free – i.e. $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r=a$

The calculation can be done easily in MATLAB:

srr =
$$-\frac{s_0\left(a^2-r^2\right)\left(r^2\cos(2\theta)-3a^2\cos(2\theta)+r^2\right)}{2r^4}$$

sqq =
$$\frac{s_0\left(\frac{2a^2}{r^2}+2\right)}{4}-\frac{s_0\cos(2\theta)\left(\frac{6a^4}{r^4}+2\right)}{4}$$

srq =
$$-\frac{s_0\sin(2\theta)\left(-3a^4+2a^2r^2+r^4\right)}{2r^4}$$

Substituting $r=a$ shows that the stress does indeed satisfy the boundary conditions.

**[3 POINTS]**

4.2 Show that the stress at $r/a \to \infty$ is $\sigma_{rr} = \sigma_0(1 + \cos(2\theta))/2$, $\sigma_{\theta\theta} = \sigma_0(1 - \cos(2\theta))/2$ $\sigma_{r\theta} = -\sigma_0 \sin(2\theta)/2$.

Substituting $a=0$ into the formulas above gives these results....

**[1 POINT]**

4.3 Show that the stresses in 4.2 are equivalent to a stress $\sigma_{11} = \sigma_0$, $\sigma_{22} = \sigma_{12} = 0$. It is easiest to work backwards – start with the stress components in the $\{e_1, e_2, e_3\}$ basis and use the basis change formulas to find the stresses in the $\{e_r, e_\theta, e_z\}$ basis

You can do this many different ways – eg use the basis change formulas with the Q matrix. Here we'll use dyadic notation.

Note that we can write the $\{e_1, e_2\}$ basis vectors as
$$e_1 = e_r \cos\theta - e_\theta \sin\theta$$
$$e_2 = e_r \sin\theta + e_\theta \cos\theta$$
Therefore
$$\sigma = \sigma_0 e_1 \otimes e_1 = \sigma_0 (e_r \cos\theta - e_\theta \sin\theta) \otimes (e_r \cos\theta - e_\theta \sin\theta)$$
$$= \sigma_0 \cos^2\theta e_r \otimes e_r + \sigma_0 \sin^2\theta e_\theta \otimes e_\theta - \sigma_0 \sin\theta \cos\theta (e_\theta \otimes e_r + e_r \otimes e_\theta)$$
(remember the dyadic product is telling us which part of the matrix each term represents - $e_1 \otimes e_1$ is the 11 term, similarly $e_r \otimes e_r$ is the $rr$ term in the polar form, and so on). We can write the answer in matrix form as

$$\sigma_0 \begin{bmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

And finally $\cos^2\theta = (1 + \cos 2\theta)/2$    $\sin^2\theta = (1 - \cos 2\theta)/2$    $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

so we get the answer we wanted.

**[3 POINTS]**

4.4 Plot a graph showing the variation of hoop stress $\sigma_{\theta\theta}/\sigma_0$ with $\theta$ at $r = a$ (the surface of the hole). What is the value of the maximum stress, and where does it occur?
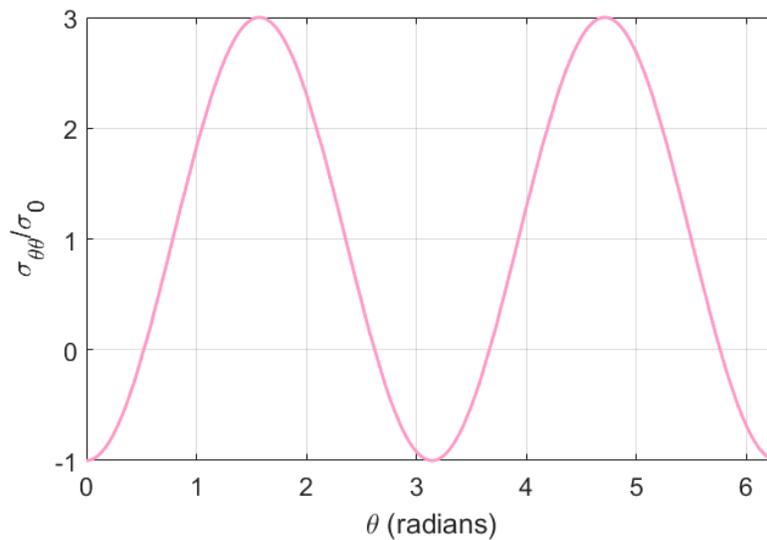
The plot is shown, the max stress occurs at the top and bottom of the hole, and it is 3 times the remote stress – so we see the famous factor of 3 for the stress concentration associated with a circular hole in an infinite plate….

**[3 POINTS]**

**5.** The picture shows a solid cylindrical column with cross-sectional area $A$ and height $L$. The column is subjected to a gravitational body force $-\rho g \mathbf{e}_2$ per unit mass.

Assume a kinematically admissible displacement field of the form

$$u_2 = \lambda_1 x_2 + \lambda_2 x_2^2$$
$$u_1 = \lambda_3 x_1 + \lambda_4 x_1 x_2$$
$$u_3 = \lambda_3 x_3 + \lambda_4 x_3 x_2$$

5.1 Use Matlab (or whatever symbolic manipulation program you prefer) to calculate the potential energy, and then minimize the potential energy with respect to $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Hence, estimate the strain and stress field in the solid.
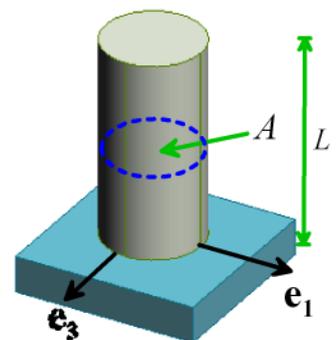


The potential energy is $\displaystyle \int_V U dV - \int_V \mathbf{b} \cdot \mathbf{u} dV$

Where $U$ is the strain energy density. For this problem we have the following strain distribution

$$\varepsilon_{11} = \varepsilon_{33} = \lambda_3 + \lambda_4 x_2 \qquad \varepsilon_{22} = \lambda_1 + 2\lambda_2 x_2$$
$$\varepsilon_{12} = \lambda_4 x_1 / 2 \qquad \varepsilon_{23} = \lambda_4 x_3 / 2 \qquad \varepsilon_{13} = 0$$

The strain energy density follows as

$$U = \frac{E}{2(1+v)}\left\{ \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2\left(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2\right) + \frac{v}{1-2v}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right)^2 \right\}$$

The total potential energy is therefore

$$\int_0^{2\pi}\int_0^{a}\int_0^{L} U r dx_2 dr d\theta + \int_0^{L} \pi a^2 \rho g(\lambda_1 x_2 + \lambda_2 x_2^2) dx_2$$

We then have to minimize wrt each of the $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. MATLAB gives

```
e11 =
```
$$-\frac{Lgv\rho\,(4Lx_2 - 4L^2 v + 3a^2 v - 4L^2 - 3a^2 + 4Lvx_2)}{2E\,(2L^2 v - 3a^2 v + 2L^2 + 3a^2)}$$

```
e22 =
```
$$-\frac{g\rho\,(2L^3 v + 2L^3 - 2x_2 L^2 v - 2x_2 L^2 - 3La^2 v^2 - 3La^2 v + 3La^2 + 6x_2 a^2 v^2 + 3x_2 a^2 v - 3x_2 a^2)}{E\,(2L^2 v - 3a^2 v + 2L^2 + 3a^2)}$$

```
e33 =
```
$$-\frac{Lgv\rho\,(4Lx_2 - 4L^2 v + 3a^2 v - 4L^2 - 3a^2 + 4Lvx_2)}{2E\,(2L^2 v - 3a^2 v + 2L^2 + 3a^2)}$$

```
e12 =
```
$$-\frac{L^2 gv\rho x_1\,(v+1)}{E\,(2L^2 v - 3a^2 v + 2L^2 + 3a^2)}$$

```
e13 = 0
e23 =
```
$$-\frac{L^2 gv\rho x_3\,(v+1)}{E\,(2L^2 v - 3a^2 v + 2L^2 + 3a^2)}$$

```
s22 =
```
$-g\rho\,(L - x_2)$

```
s11 =
```
$$-\frac{3a^2 gv\rho\,(L - 2x_2)}{4L^2 v - 6a^2 v + 4L^2 + 6a^2}$$

```
s33 =
```
$$-\frac{3a^2 gv\rho\,(L - 2x_2)}{4L^2 v - 6a^2 v + 4L^2 + 6a^2}$$

```
s12 =
```
$$-\frac{L^2 gv\rho x_1}{2L^2 v - 3a^2 v + 2L^2 + 3a^2}$$

```
s32 =
```
$$-\frac{L^2 gv\rho x_1}{2L^2 v - 3a^2 v + 2L^2 + 3a^2}$$

MATLAB code is below

```
clear all
syms lambda_1 lambda_2 lambda_3 lambda_4 x1 x2 x3 r q L rho g E nu a real
u2 = simplify(lambda_1*x2 + lambda_2*x2^2);
u1 = simplify(lambda_3*x1 + lambda_4*x1*x2);
u3 = simplify(lambda_3*x3 + lambda_4*x3*x2);
e11 = diff(u1,x1);
e22 = diff(u2,x2);
e33 = diff(u3,x3);
e12 = (diff(u1,x2) + diff(u2,x1))/2;
e13 = (diff(u1,x3) + diff(u3,x1))/2;
e23 = (diff(u2,x3) + diff(u3,x2))/2;
U = (1/2)*E/(1+nu)*( e11^2 + e22^2 + e33^2 + 2*e12^2 + 2*e13^2 + 2*e23^2 + ...
    nu/(1-2*nu)*(e11+e22+e33)^2 );
U = subs(U,[x1,x3],[r*cos(q),r*sin(q)]);
PE = int(int(int(U*r,x2,[0,L]),q,[0,2*pi]),r,[0,a]) + pi*a^2*int(rho*g*(lambda_
eq1 = diff(PE,lambda_1) == 0;
eq2 = diff(PE,lambda_2) == 0;
eq3 = diff(PE,lambda_3) == 0;
eq4 = diff(PE,lambda_4) == 0;
[lambda_1,lambda_2,lambda_3,lambda_4] = solve([eq1,eq2,eq3,eq4],[lambda_1,lambd
u2 = simplify(lambda_1*x2 + lambda_2*x2^2)
u1 = simplify(lambda_3*x1 + lambda_4*x1*x2)
u3 = simplify(lambda_3*x3 + lambda_4*x3*x2);
e11 = simplify(diff(u1,x1))
e22 = simplify(diff(u2,x2))
e33 = simplify(diff(u3,x3))
e12 = simplify((diff(u1,x2) + diff(u2,x1))/2)
e13 = simplify((diff(u1,x3) + diff(u3,x1))/2)
e23 = simplify((diff(u2,x3) + diff(u3,x2))/2)
s22 = simplify(E/(1+nu)/(1-2*nu)*( (1-nu)*e22 + 2*nu*e11 ))
s11 = simplify(E/(1+nu)/(1-2*nu)*( (1-nu)*e11 + nu*(e22 +e33 )) )
s33 = simplify(E/(1+nu)/(1-2*nu)*( (1-nu)*e33 + nu*(e11 +e22 )) )
s12 = simplify(E/(1+nu)*e12 )
s32 = simplify(E/(1+nu)*e12 )
```

This is not strictly legal, but you can get a simpler approximation if you ignore the contribution to the strain energy density from the shear strains.

$$U \approx \frac{E}{2(1+v)} \left\{ \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \frac{v}{1-2v}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})^2 \right\}$$

In which case

ell =

$$\frac{g v \rho (L - x_2)}{E}$$

e22 =

$$-\frac{g \rho (L - x_2)}{E}$$

e33 =

$$\frac{g v \rho (L - x_2)}{E}$$

$$s22 = -g \rho (L - x_2)$$

$$s11 = 0$$

$$s33 = 0$$

This is what you would calculate if you assumed uniaxial stress, and solved the problem by hand.   This solution satisfies equilibrium, but the strain distribution is not compatible.

If people used this solution it should get credit too, even though it is not quite right.

**[4 POINTS]**

5.2 Is the solution obtained in this way exact?  (i.e. does the stress field satisfy equilibrium and boundary conditions?)

The solution is not exact, because it doesn't satisfy the boundary conditions on the sides of the bar, and it doesn't satisfy equibrium exactly either.

The solution does become more and more accurate as the cylinder gets longer compared with its radius, however.

**[1 POINT]**
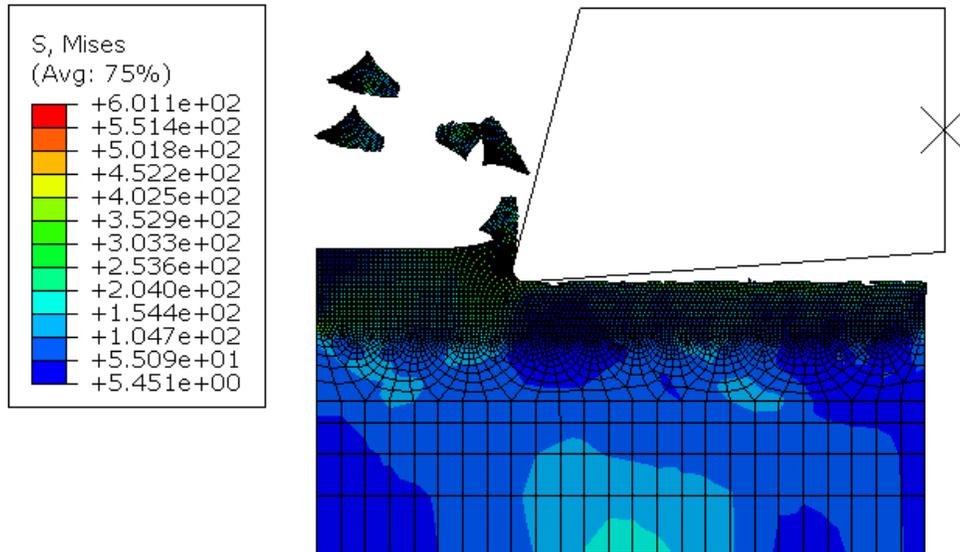
**6. Orthogonal machining model**: In this problem you will set up a simple 2D model of an orthogonal machining process (see the figure). The problem uses several advanced techniques in FEA:

(1) Modeling contact

(2) Modeling material failure, with element deletion

(3) Using the 'Arbitrary Lagrangean-Eulerian' formulation available in ABAQUS to avoid problems caused by large distortion of the mesh.

The goal is to set up the model shown in the figure: a thin layer of material is removed from a workpiece (made from Al) by a rigid cutting tool.

For a unit system we will work with dimensions in mm; forces in N; stresses in MPa, and time in milliseconds (note that 1 m/s is 1 mm/millisecond).

6.1 Start by creating parts for both the workpiece and cutting tool. The workpiece is a 2D deformable part; the tool is a discrete rigid part with Wireframe base feature. Relevant dimensions are shown in the figure. To make it easier to position the parts in the assembly, place the top right corner of the rectangle representing the workpiece at the origin. You don't have to get the tool dimensions exactly the same as those shown in the figure. Note that the dimensions are in mm.
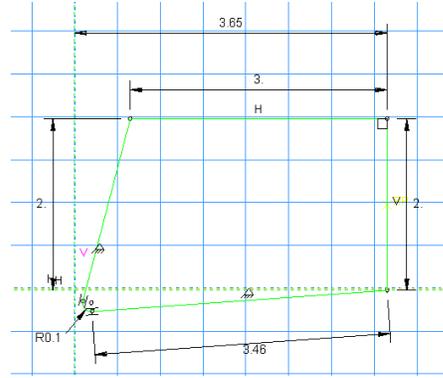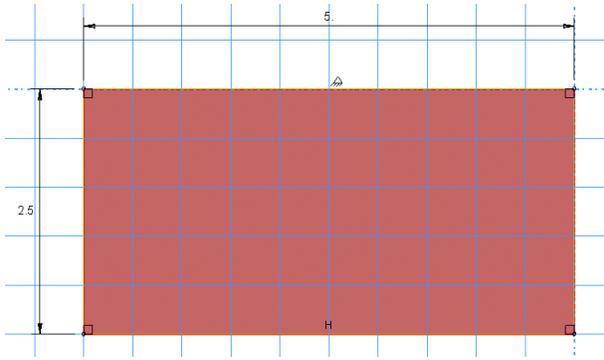
For mesh generation and adaptive meshing we will have to partition the workpiece as shown below. To do this, first partition the block down the middle – select the 'partition by shortest distance between two points'  tool from the toolbar then select the midpoints on the two vertical sides of the workpiece. Next, partition the top half of the block down the middle. To do this select the 'Partition face by sketch'  tool, select the top half of the block in the viewport and press 'Done' then sketch a horizontal line down the center of the top half of the block. You should end up with a partitioned part that looks like the picture shown below.
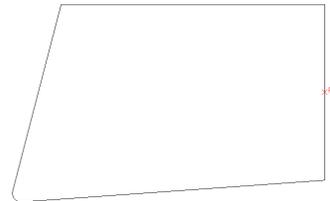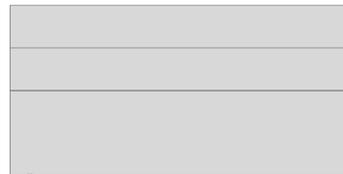


Finally we need to add a reference point so we can apply boundary conditions to the tool. Make sure the tool is shown in the viewport, then select Tools > Reference point, and select the point midway down the back face of the tool (see the figure above).

6.2 In the Property module, create a new material called Al that will be assigned to the workpiece. This publication has material properties for a Johnson-Cook plasticity/damage model for a 5xxx series Al alloy. Enter values in MPa (this is convenient because $1 N/mm^2$ is the same as 1 MPa, so forces will be in Newtons)

- Young's modulus $E = 70\,000$ MPa
- Poissons ratio 0.3
- Density: 2700 $kg/m^3$ – but we have to be careful to enter its value in a consistent set of units. Convince yourself that if we choose to use N for force, mm for length, and milliseconds for time, we must enter density in N milliseconds$^2$/mm$^4$, which makes $\rho = 2.7 \times 10^{-3}$ in our chosen unit system
- Plasticity – (select Mechanical>Plastic and use the Johnson-Cook hardening model in the drop-down menu. Then enter values of the parameters that define how the material hardens with strain and strain rate:
  A=277.3MPa, B=307.93MPa, n=0.69, m=0, Melting T=0, Transition T=0 (temp is neglected)
  Select Suboptions, then select 'Rate dependent' and select the Johnson-Cook suboption. Enter C=0.0032,and Epsilon dot zero = $10^{-6}$ (in milliseconds$^{-1}$)

- Next, enter parameters that will control how the material begins to fail. In ABAQUS failure is modeled by making the material soften once the strain exceeds a critical value. To enter the parameters select

  Damage – select Mechanical > Damage for ductile metals > Johnson-Cook damage and enter for the constants:

  $d1_1$=0.211, d2=3.9, d3= -4.7, d4=d5=0

  Finally, we need to enter a parameter that ABAQUS uses to control how elements are removed from the mesh after they fail. This is really a correction factor used during the simulation rather than a material property. It has two purposes: one is to make sure that the stress is reduced to zero gradually after an element fails; the second is to reduce sensitivity to mesh size that occurs in softening materials.

  To do this select Suboption > Damage Evolution, and enter 0.01mm for the 'displacement at failure' (this makes ABAQUS remove an element when the plastic strain multiplied by the element size reaches 0.01mm)

(Read the publication cited or the ABAQUS theory manual for definitions of the constants)

Finally, create a homogeneous plane section with plane strain thickness of 5mm and assign it to the workpiece (the block).

6.3 Create an instance of both parts in the Assembly module.

6.4 In the Step module, create and Explicit Dynamic step with duration 17 milliseconds (enter 17 in the window, not 0.017, since we are working with millisecond time units). In the 'Mass Scaling' tab, check the radio button for 'Use scaling definitions below' and use Create… to define 'mass scaling' for the step. The purpose mass scaling is to increase the stable time-step for the analysis (we will discuss this in more detail later in the course). In the Edit Mass Scaling menu check the 'Scale By Factor' box and enter 10000 for the mass scaling (this is a rather too large –, but with smaller values you have to wait forever for the job to run).

Secondly, make sure the 'Status' variable is saved to the output database (otherwise elements don't get deleted and the deformed mesh looks extremely scary). To do this select Output > Field Output Requests > Manager, then edit F-output-1. Expand the State/Field/User/Time list of variables and check the Status box and press OK at the bottom of the editor.

☐ DENSITYVAVG, Volume-averaged material density (Eulerian on
☑ STATUS, Status (some failure and plasticity models; VUMAT)
☐ EACTIVE, Status of the element

Next, use Tools > Set > Create to create a new set called Tool-Refpoint, and select the reference point on the tool. Then use Output> History Output Requests> Create… to create a new History Output request that will make ABAQUS save the forces and displacements of the reference point on the tool to the output database (select 'Set' for the domain, then you can select the Tool-Refpoint set; and check the boxes in the Output Variables list). Save the request….

Finally, the step menu also gives you an option to activate ALE remeshing. To do this select Other > ALE Adaptive Mesh Domain > Manager, and click on Step-1 and select 'Edit' in the domain manager. An Edit menu should open: select the 'Use the ALE adaptive mesh domain below' radiobutton, then press the arrow next to 'Region' and select the top rectangle of the part in the viewport. Accept all the defaults and press OK.

(Optional – if you want to make a smooth animation of the cutting process you will need to select more frequent outputs to the odb than the default – save every 0.5 secs or more frequently)

6.5 In the Interaction module: (a) create an interaction property with Hard normal contact and Penalty tangential contact with friction coefficient 0.1    (b) Create an Interaction (Interaction > Create, select Surface to Surface Contact (Explicit), then below the viewport select 'by angle' in the dropdown menu, and click on the fillet radius of the tool.   The front face of the tool should be highlighted in the viewport.    Select the color corresponding to the arrow on the outer side of the tool in the menu below the viewport.   Then for the second surface select 'Node region' and select the rectangle at the top of the workpiece.   The interaction should be highlighted as shown.  Accept the default options in the interaction definition menu.
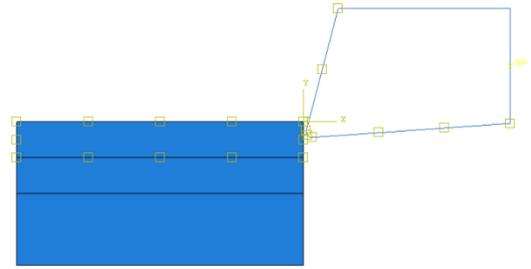


6.6 For boundary conditions: (1) Prescribe zero vertical displacement on the bottom of the block; (2) Prescribe zero horizontal displacement on the left edge of the bottom two partitions of the block; (3) Create a 'velocity/angular velocity' boundary condition for the reference point with a horizontal velocity of -0.25 mm/ms and zero vertical and angular velocity.

6.7 For meshing, use Mesh> Element Type… to assign Explicit Plane Strain elements – **check that the 'Element Deletion' option has the 'yes' radio button checked** -  to the workpiece, then create a mesh like the one below (you can create it by seeding the various horizontal edges with different mesh densities.  In the figure shown the top part has a uniform mesh size of 0.02mm).   Seed the tool with a mesh size 0.02mm and mesh it.
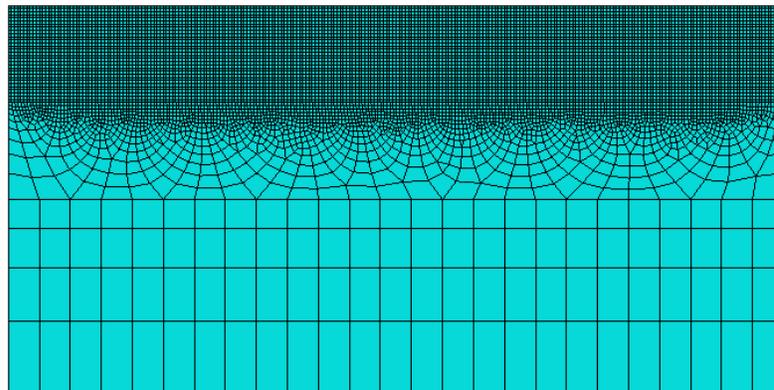


6.8 Create a job and run it – it will take some time to complete – you can use Job> Monitor to track its progress.

As a solution to this problem create a plot of the horizontal force acting on the tool as a function of time.

**[10 POINTS]**