School of Engineering
Brown University

EN1750: Advanced Mechanics of Solids

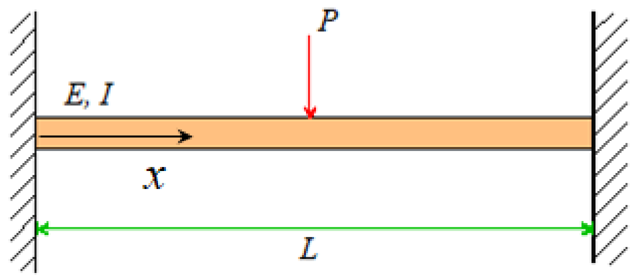
Homework 6: Energy methods, Implementing FEA.

Due Friday Nov 9, 2018

1. The figure shows a beam with clamped ends subjected to a point force at its center. Its potential energy is

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v(x)}{dx^2} \right)^2 dx - P v(L/2)$$

Where v is the downward deflection of the beam.



1.1 Start by re-writing the potential energy in dimensionless form, by defining dimensionless measures of deflection and position as follows

$$\hat{v} = \frac{vEI}{PL^3} \qquad \hat{x} = \frac{x}{L} \qquad \hat{\Pi} = \frac{\Pi EI}{P^2 L^3}$$

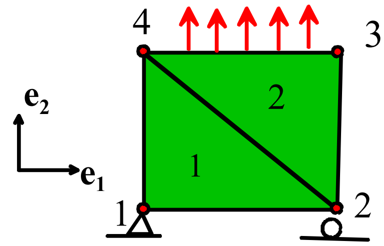
Show that the dimensionless form for potential energy is independent of loading, geometry, or material properties (so you don't need to know values for these parameters to set up a MATLAB code).

1.2 Use a displacement field of the form

$$\hat{v}(x) = \sum_{n=1}^{N} a_n \hat{x}^{n-1}$$

to obtain a Rayleigh-Ritz approximation to the deflected shape of the beam. Write a MATLAB script (you can modify the one from class – you will only need to change a few lines in the code) to calculate $\hat{v}(\hat{x})$ for an arbitrary number of terms in the approximation. Plot a graph of the solution with 5, 10 and 15 terms in the series. Hand in a copy of your plot, there is no need to submit MATLAB code

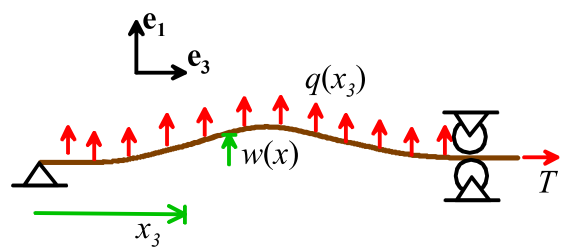
2. Modify the Matlab code given in class to solve problems involving *plane stress* deformation instead of plane strain (this should require a change to only one line of the code).



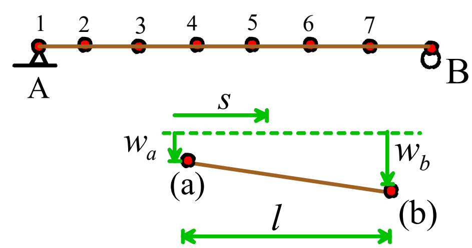
2.1 Check the modified code by solving the problem shown in the figure. Assume that the block has unit length in both horizontal and vertical directions, use Young's modulus 100 and Poisson's ratio 0.3, and take the magnitude of the distributed load to be 10 (all in arbitrary units). Compare the predictions of the FEA analysis with the exact solution (report your comparison as a table that gives your FEA predictions for the displacements at the 4 nodes, along with the exact solution). Please hand in a description of the line(s) of code you modified as a solution to this problem, and the table. There is no need to submit MATLAB code.

2.2 Repeat the test done in class to calculate the stress field in plate containing a central hole, with Poissons ratio 0.499 (you can download a copy of the input file for this simulation of the homework page of the website). Plot the stress contours, and find the predicted maximum stress concentration factor (you can just search for the element with the maximum stress). Compare the solution with your hand calculations using the Airy function in Homework 5. Upload a copy of your contour plot along with the hand calculation. There is no need to submit MATLAB code.

3. In this problem you will develop and apply a finite element method to calculate the shape of a tensioned, inextensible cable subjected to transverse loading (e.g. gravity or wind loading). The cable is pinned at A, and passes over a frictionless pulley at B. A tension T is applied to the end of the cable as shown. A (nonuniform) distributed load $q(x)$ causes the cable to deflect by a distance $w(x)$ as shown. For w<<L, the potential energy of the system may be approximated as



$$\Pi = \int_0^L \frac{T}{2}\left(\frac{dw}{dx}\right)^2 dx - \int_0^L qw dx$$



To develop a finite element scheme to calculate w, divide the cable into a series of 1-D finite elements as shown. Consider a generic element of length l with nodes a, b at its ends. Assume that the load q is uniform over the element, and assume that w varies linearly between values w_a, w_b at the two nodes.

3.1 Write down an expression for w at an arbitrary distance s from node a, in terms of w_a, w_b, s and l. (use a linear interpolation)

3.2 Deduce an expression for dw/dx within the element, in terms of w_a, w_b and l

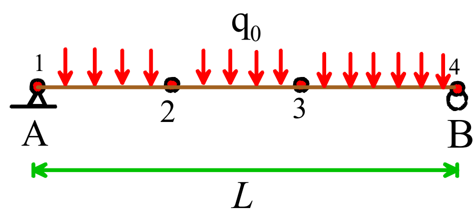
3.3 Hence, calculate an expression for the contribution to the potential energy arising from the element shown, and show that element contribution to the potential energy may be expressed as

$$\Pi^{\text{elem}} = \frac{1}{2}\left[w_a, w_b\right]\begin{bmatrix} T/l & -T/l \\ -T/l & T/l \end{bmatrix}\begin{bmatrix} w_a \\ w_b \end{bmatrix} - \left[w_a, w_b\right]\begin{bmatrix} ql/2 \\ ql/2 \end{bmatrix}$$

(assume that q is uniform over the segment)

3.4 Write down expressions for the element stiffness matrix and force vector (just read them off the formula for the potential energy).

3.5 Consider the finite element mesh shown in the figure. The loading q_0 is uniform, and each element has the same length. The cable tension is T.



Use the procedure discussed in class to write down the total potential energy (obtained by summing up contributions from the 3 elements) in terms of the global displacement vector $[w_1, w_2, w_3, w_4]$ Hence, find formulas for the 4x4 global stiffness matrix and 4x1 residual vector for the mesh, in terms of *T*, *L*, and q_0 (before any constraints are applied).

3.6 Show how the global stiffness matrix and residual vectors must be modified to enforce the constraints $w_1 = w_4 = 0$

3.7 Hence, calculate values of w at the two intermediate nodes.

3.8 Write a simple MATLAB script to solve the problem for an arbitrary number of elements with equal spacing between nodes (you can work out what the stiffness matrix and residual vector will look like for an arbitrary number of nodes from the pattern you see in 3.6 - you don't need to write a fancy code that will assemble the matrix from scratch, although you are more than welcome to do so if you would like to work through this).

Run your code with some sensible values for *T*, *L*, and q_0, and compare the FEA predictions with the exact solution.

$$w = \frac{q_0}{2T} x_3 (L - x_3)$$

Please upload your MATLAB code as a solution to this problem; make your code plot a graph comparing the exact and FEA solutions.