School of Engineering
Brown University

EN1750: Advanced Mechanics of Solids

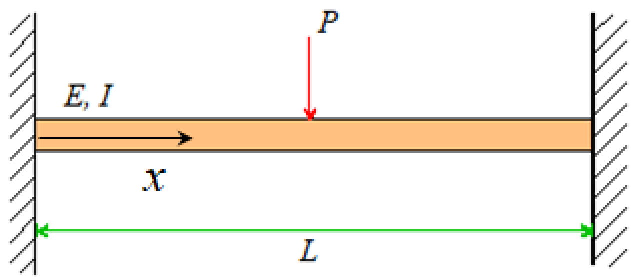
Homework 6: Energy methods, Implementing FEA.

Due Friday Nov 9, 2018

1. The figure shows a beam with clamped ends subjected to a point force at its center. Its potential energy is

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2 v(x)}{dx^2} \right)^2 dx - Pv(L/2)$$

Where v is the downward deflection of the beam.



1.1 Start by re-writing the potential energy in dimensionless form, by defining dimensionless measures of deflection and position as follows

$$\hat{v} = \frac{vEI}{PL^3} \qquad \hat{x} = \frac{x}{L} \qquad \hat{\Pi} = \frac{\Pi EI}{P^2 L^3}$$

Show that the dimensionless form for potential energy is independent of loading, geometry, or material properties.

Substituting for the relevant variables gives

$$\hat{\Pi} \frac{P^2 L^3}{EI} = \int_0^1 \frac{1}{2} EI \left(\frac{PL^3}{EIL^2} \frac{d^2 \hat{v}(x)}{d\hat{x}^2} \right)^2 L d\hat{x} - P \frac{PL^3}{EI} \hat{v}(L/2)$$

$$\Rightarrow \hat{\Pi} = \int_0^1 \frac{1}{2} \left(\frac{d^2 \hat{v}(x)}{d\hat{x}^2} \right)^2 d\hat{x} - \hat{v}(L/2)$$

The normalized expression has no parameters so it is universal.

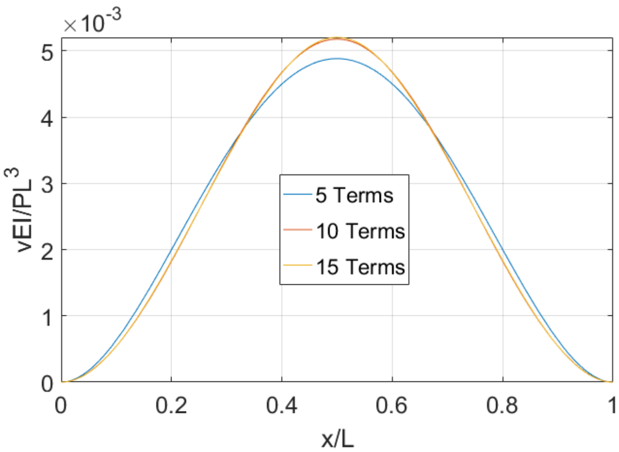
[2 POINTS]

1.2 Use a displacement field of the form

$$\hat{v}(x) = \sum_{n=1}^{N} a_n \hat{x}^{n-1}$$

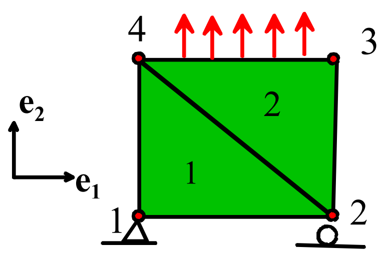
to obtain a Rayleigh-Ritz approximation to the deflected shape of the beam. Write a MATLAB script (you can modify the one from class – you will only need to change a few lines in the code) to calculate $\hat{v}(\hat{x})$ for an arbitrary number of terms in the approximation. Plot a graph of the solution with 5, 10 and 15 terms in the series. Hand in a copy of your plot, there is no need to submit MATLAB code

The plot is shown below



[5 POINTS]

2. Modify the Matlab code given in class to solve problems involving *plane stress* deformation instead of plane strain (this should require a change to only one line of the code).



2.1 Check the modified code by solving the problem shown in the figure. Assume that the block has unit length in both horizontal and vertical directions, use Young's modulus 100 and Poisson's ratio 0.3, and take the magnitude of the distributed load to be 10 (all in arbitrary units). Compare the predictions of the FEA analysis with the exact solution (report your comparison as a table that gives your FEA predictions for the displacements at the 4 nodes, along with the exact solution). Please hand in a description of the line(s) of code you modified as a solution to this problem, and the table. There is no need to submit MATLAB code.

To do this you need to

(1) Change the elastic constant matrix (line 24) to

```
Dmat = [[1,nu,0];[nu,1,0];[0,0,(1-nu)/2]]*E/((1-nu^2));
```

(2) Edit the input file to enforce the correct boundary conditions

```
Material_Props:
    Young's_modulus:   100.
    Poissons_ratio:  0.3
No._nodes:            4
Nodal_coords:
    0.0   0.0
    1.0   0.0
    0.0   1.0
    1.0   1.0
No._elements:                   2
Element_connectivity:
    1 2 3
    2 4 3
No._nodes_with_prescribed_DOFs:  3
Node_#, DOF#, Value:
   1 1 0.0
   1 2 0.0
   2 2 0.0
No._elements_with_prescribed_loads: 1
Element_#, Face_#, Traction_components
  2 2 0.0 10.0
```

The predicted displacement field is below

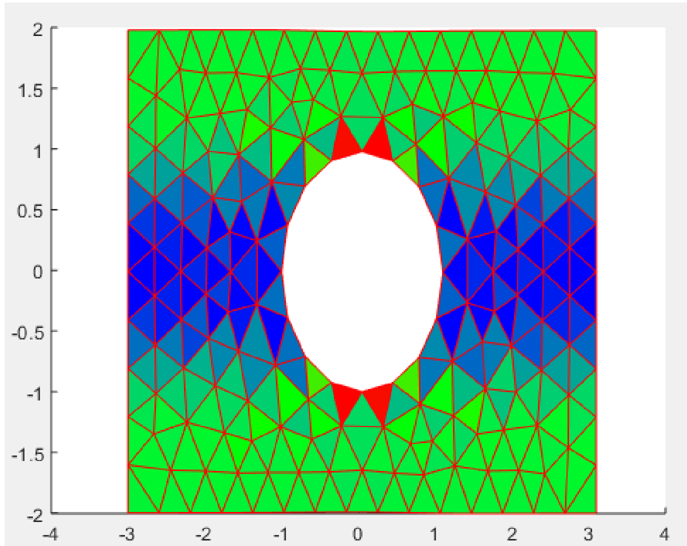
```
Nodal Displacements:
 Node      u1          u2
  1   -0.0000    0.0000
  2   -0.0300    0.0000
  3    0.0000    0.1000
  4   -0.0300    0.1000
```

To get the exact solution note that the material is in a state of uniaxial tension; the strain is therefore 10/100=0.1; the vertical displacement of the top will be 0.1; the lateral displacement of nodes 2 and 4 will be $-\nu\varepsilon_{22}L = -0.3$. So the FEA solution is exact.

[3 POINTS]

2.2 Repeat the test done in class to calculate the stress field in plate containing a central hole, with Poissons ratio 0.499 (you can download a copy of the input file for this simulation of the homework page of the website). Plot the stress contours, and find the predicted maximum stress concentration factor (you can just search for the element with the maximum stress). Compare the solution with your hand calculations using the Airy function in Homework 5. Upload a copy of your contour plot along with the hand calculation. There is no need to submit MATLAB code.

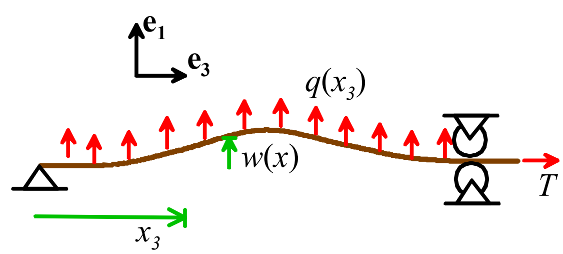
The stress contours are shown. There is no locking with plane stress elements.



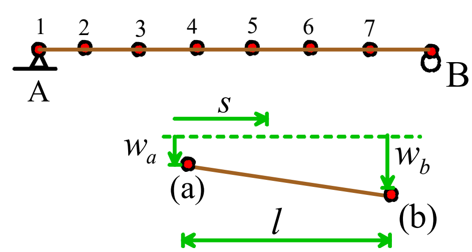
Matlab reports the maximum stress as 4.11 units. The remote stress can be calculated from the displacement applied to the boundary (0.1), which gives a strain of 0.1/6; the modulus is 100 so the remote stress is 10/6. This gives a stress concentration of 2.5 - a bit lower than the analytical solution predicts, but the mesh is extremely coarse….

[2 POINTS]

3. In this problem you will develop and apply a finite element method to calculate the shape of a tensioned, inextensible cable subjected to transverse loading (e.g. gravity or wind loading). The cable is pinned at A, and passes over a frictionless pulley at B. A tension T is applied to the end of the cable as shown. A (nonuniform) distributed load $q(x)$ causes the cable to deflect by a distance $w(x)$ as shown. For w<<L, the potential energy of the system may be approximated as



$$\Pi = \int_0^L \frac{T}{2}\left(\frac{dw}{dx}\right)^2 dx - \int_0^L qw dx$$



To develop a finite element scheme to calculate w, divide the cable into a series of 1-D finite elements as shown. Consider a generic element of length l with nodes a, b at its ends. Assume that the load q is uniform over the element, and assume that w varies linearly between values w_a, w_b at the two nodes.

3.1 Write down an expression for w at an arbitrary distance s from node a, in terms of w_a, w_b, s and l. (assume a linear variation)

$$w = w_a(1 - s/l) + w_b s/l$$

[2 POINTS]

3.2 Deduce an expression for dw/dx within the element, in terms of w_a, w_b and l

$$dw/dx = (w_b - w_a)/l$$

[1 POINT]

3.3 Hence, calculate an expression for the contribution to the potential energy arising from the element shown, and show that element contribution to the potential energy may be expressed as

$$V^{\text{elem}} = \frac{1}{2}\left[w_a, w_b\right]\begin{bmatrix} T/l & -T/l \\ -T/l & T/l \end{bmatrix}\begin{bmatrix} w_a \\ w_b \end{bmatrix} - \left[w_a, w_b\right]\begin{bmatrix} ql/2 \\ ql/2 \end{bmatrix}$$

(assume that q is uniform over the segment)

We can compute the contribution from one element to the two terms in the potential energy

$$\int_0^l \frac{T}{2}\left(\frac{dw}{dx}\right)^2 dx = \frac{T}{2l}(w_b - w_a)^2 = \frac{T}{2l} w_a(w_a - w_b) + \frac{T}{2l} w_b(w_b - w_a) = \left[w_a, w_b\right]\begin{bmatrix} T/l & -T/l \\ -T/l & T/l \end{bmatrix}\begin{bmatrix} w_a \\ w_b \end{bmatrix}$$

$$-\int_0^l qw dx = -\frac{qL}{2}(w_a + w_b) = -\left[w_a, w_b\right]\begin{bmatrix} ql/2 \\ ql/2 \end{bmatrix}$$

[3 POINTS]

3.4 Write down expressions for the element stiffness matrix and force vector.

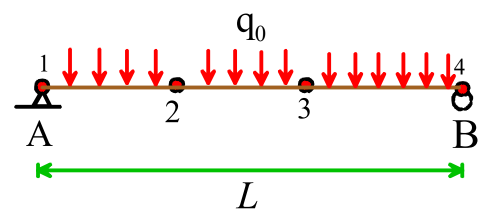
By inspection

$$[k_{el}] = \begin{bmatrix} T/l & -T/l \\ -T/l & T/l \end{bmatrix}$$

$$[r_{el}] = \begin{bmatrix} ql/2 \\ ql/2 \end{bmatrix}$$

[1 POINT]

3.5 Consider the finite element mesh shown in the figure. The loading q_0 is uniform, and each element has the same length. The cable tension is T. Calculate the global stiffness matrix and residual vectors for the mesh, in terms of T, L, and q_0 (before any constraints are applied).



Following the procedure discussed in class, we add contributions from neighboring elements to each node. The two central nodes are shared by two elements, so

$$[K] = \frac{T}{L}\begin{bmatrix} 3 & -3 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\underline{r} = \frac{q_0 L}{6}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

[2 POINTS]

3.6 Show how the global stiffness matrix and residual vectors must be modified to enforce the constraints $w_1 = w_4 = 0$

We need to replace the first and last equations to enter the two constraint equations

$$[K] = \frac{T}{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{r} = \frac{q_0 L}{3}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

[2 POINTS]

3.7 Hence, calculate values of w at the two intermediate nodes.

The equation system can be solved to give $w_2 = w_3 = \dfrac{q_0 L^2}{9T}$

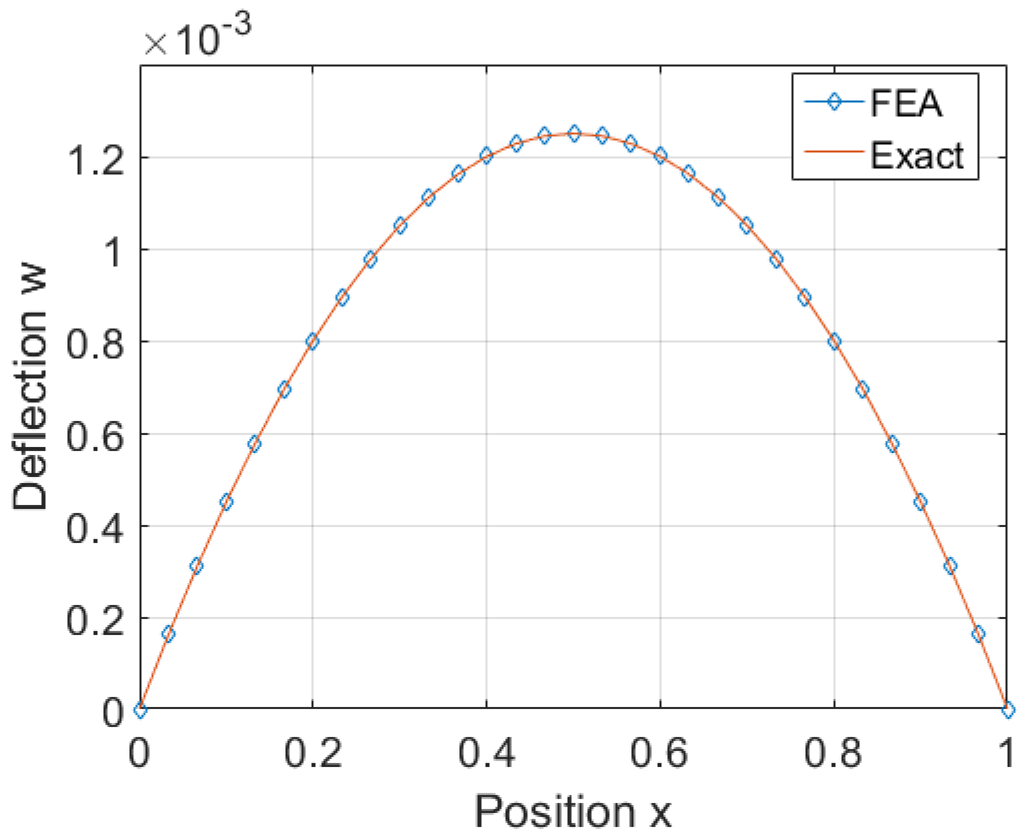
[2 POINTS]

3.8 Write a simple MATLAB script to solve the problem for an arbitrary number of elements with equal spacing between nodes (you can just use the pattern you see in 3.6; you don't need to write a fancy code). Compare the FEA predictions with the exact solution.

$$w = \frac{q_0}{2T} x_3 (L - x_3)$$

Please upload your MATLAB code as a solution to this problem; make your code plot a graph comparing the exact and FEA solutions.

The plot is shown below



The FEA solution (at the nodes) turns out to be exact regardless of the number of elements!

[5 POINTS]