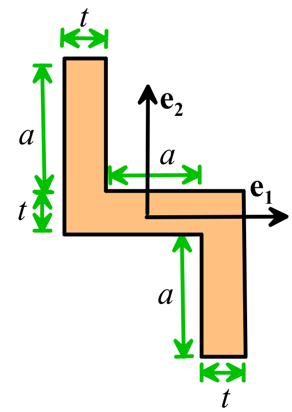
School of Engineering
Brown University

**EN1750: Advanced Mechanics of Solids**

**Homework 7: Strings, Beams, Membranes and Plates.**

Due Friday Nov 16, 2018

**1.** The figure shows the cross-section of a beam.




1.1 Calculate the area moment of inertia tensor for the beam section shown (simplify the solution by assuming t<<a) (you can use the matlab script from class).

A Matlab solution is shown below.

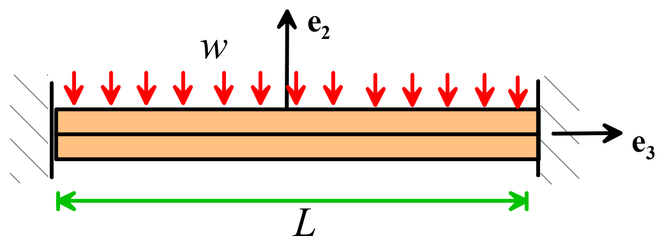
```
clear all
syms r rbar a t A II x1 x2
% Compute the area
A = int(int(1,x1,[-a/2-t,a/2+t]),x2,[-t/2,t/2]) +...
    int(int(1,x1,[-a/2-t,-a/2]),x2,[t/2,t/2+a]) + ...
    int(int(1,x1,[a/2,a/2+t]),x2,[-t/2-a,-t/2])
r = [x1,x2];
% Find the centroid (neutral line)
rbar = int(int(r,x1,[-a/2-t,a/2+t]),x2,[-t/2,t/2]) +...
    int(int(r,x1,[-a/2-t,-a/2]),x2,[t/2,t/2+a]) + ...
    int(int(r,x1,[a/2,a/2+t]),x2,[-t/2-a,-t/2])
% Calculate the inertia tensor
integrand = [(r(2)-rbar(2))^2, -(r(1)-rbar(1))*(r(2)-rbar(2));...
             -(r(1)-rbar(1))*(r(2)-rbar(2)),(r(1)-rbar(1))^2];
II = simplify(int(int(integrand,x1,[-a/2-t,a/2+t]),x2,[-t/2,t/2]) +...
    int(int(integrand,x1,[-a/2-t,-a/2]),x2,[t/2,t/2+a]) + ...
    int(int(integrand,x1,[a/2,a/2+t]),x2,[-t/2-a,-t/2]))
%  Find the directions of the principal axes of inertia (colums of V), and
%  the principal moments of inertia (D)
[V,D] = eig(II)
% Get approximate solutions for small t
taylor(II,t,'Order',2)
taylor(D,t,'Order',2)
```

```
ans =
```

$$\begin{pmatrix} \frac{2a^3 t}{3} & \frac{a^3 t}{2} \\ \frac{a^3 t}{2} & \frac{7a^3 t}{12} \end{pmatrix}$$

**[5 POINTS]**

1.2 Suppose that a beam with length $L$ and cross-section analyzed in 1.1 is subjected to a uniform load $w$ acting in the (negative) $\mathbf{e}_2$ direction. Calculate the deflection of the beam at mid-span (you can use the Matlab script from class)




The governing equations are

$$E\left( I_{22}\frac{d^4u_1}{dx_3^4} + I_{12}\frac{d^4u_2}{dx_3^4} \right) = 0$$

$$E\left( I_{12}\frac{d^4u_1}{dx_3^4} + I_{11}\frac{d^4u_2}{dx_3^4} \right) = -w$$

The boundary conditions are zero displacement and slope at both ends of the beam.

It's easy to solve the equations with MATLAB

```
% Script to calculate deflections of a beam
% You can easily adapt this script to other loads and boundary conditions
clear all
syms EE I11 I22 I12 II M T1 T2 T3 a t L P x3 C1 C2 C3 C4 w real
syms u1(x3) u2(x3)
% The second moments of area
I11 = 2*a^3*t/3;
I12 = -a^3*t/2; % A positive off diagonal means I12 is a negative number
I22 = 7*a^3*t/12;
% The two differential equations
diffeq1 = EE*(I22*diff(u1(x3),x3,4)+I12*diff(u2(x3),x3,4))==0;
diffeq2 = EE*(I12*diff(u1(x3),x3,4)+I11*diff(u2(x3),x3,4))==-w;
% Zero displacement and slope at x3=0
BC1 = u1(0)==0;
BC2 = u2(0)==0;
BC3 = subs(diff(u1,x3),x3,0)==0;
BC5 = u1(L) == 0;
BC6 = u2(L) == 0;
BC7 = subs(diff(u1(x3),x3),x3,L)==0;
BC8 = subs(diff(u2(x3),x3),x3,L)==0;
% The solution with displacements/slopes prescribed at both ends
[u1,u2] = dsolve([diffeq1,diffeq2],[BC1,BC2,BC3,BC4,BC5,BC6,BC7,BC8]);
% Now calculate the bending moments and transverse forces
simplify(subs(u1,x3,L/2))
simplify(subs(u2,x3,L/2))
```

ans =

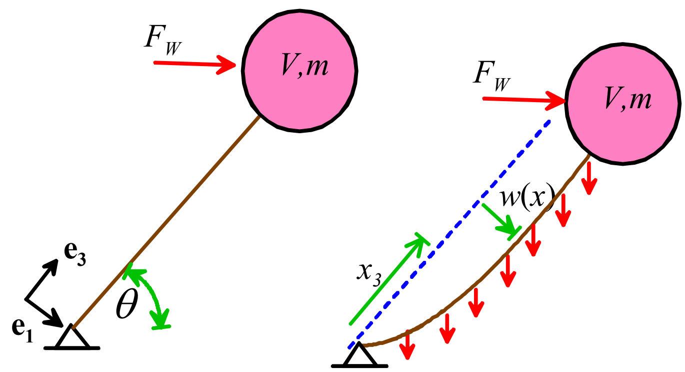
$$-\frac{3\,L^4\,w}{320\,\mathrm{EE}\,a^3\,t}$$

ans =

$$-\frac{7\,L^4\,w}{640\,\mathrm{EE}\,a^3\,t}$$

**[5 POINTS]**

**2.** The figure shows a spherical balloon with volume $V$ and mass $m$ tethered by a cable with mass $\mu$ per unit length. The balloon is subjected to a lateral wind load $F_W$




2.1 For the limiting case of a massless cable ( $\mu = 0$ ), the cable remains straight and has a constant internal tension $T$. Find formulas for the angle $\theta$ and $T$ in terms of $V, m$, $F_W$ and air density $\rho$ .

A simple static equilibrium calculation for the balloon gives

$$\theta = \tan^{-1}\frac{(\rho V - m)g}{F_W} \qquad T = \sqrt{F_W^2 + (\rho g V - mg)^2}$$

**[2 POINTS]**

2.2 If the cable mass is significant the cable will deflect as shown in the figure. Assuming that the cable is in static equilibrium, write down

- The differential equation governing the variation of tension in the cable
- The boundary condition for the tension at $x_3 = L$
- The differential equation for the deflection $w$
- The boundary conditions for $w$ at $x_3 = 0$ and $x_3 = L$

From notes:

$$\frac{dT_3}{dx_3} + p_3 \approx 0 \Rightarrow \frac{dT_3}{dx_3} - \mu g \sin\theta = 0$$

The boundary condition is $T_3 = \sqrt{F_W^2 + (\rho g V - mg)^2}$ at $x_3 = L$

$$\frac{d}{dx_3}\left(T_3 \frac{du_1}{dx_3}\right) + p_1 = 0 \Rightarrow \frac{d}{dx_3}\left(T_3 \frac{du_1}{dx_3}\right) + \mu g \cos\theta = 0$$

The boundary condition is $w = 0 \quad x_3 = 0 \qquad \frac{dw}{dx_3} = 0 \qquad x_3 = L$ (the second equation is because the tension must act on the balloon in the same direction after the cable deflects)

**[3 POINTS]**

2.3 Hence, solve the equation system in the preceding section to show that the cable deflection $w$ is

$$w = \frac{b}{c}\left\{ -x_3 + \frac{T_0}{c}\log\frac{T_0 - c(L - x_3)}{T_0 - cL} \right\}$$

$$c = \mu g \sin\theta \qquad b = \mu g \cos\theta \qquad T_0 = \sqrt{F_W^2 + (\rho g V - mg)^2}$$

We can integrate the first equation and use the boundary condition to get

$$T_3 = T_0 - c(L - x_3) \qquad c = \mu g \sin\theta \quad T_0 = \sqrt{F_W^2 + (\rho g V - mg)^2}$$

The equation for lateral deflection gives

$$\frac{d}{dx_3}\left[ (T_0 - c(L - x_3)\ )\frac{dw}{dx_3} \right] + b = 0 \qquad b = \mu g \cos\theta$$

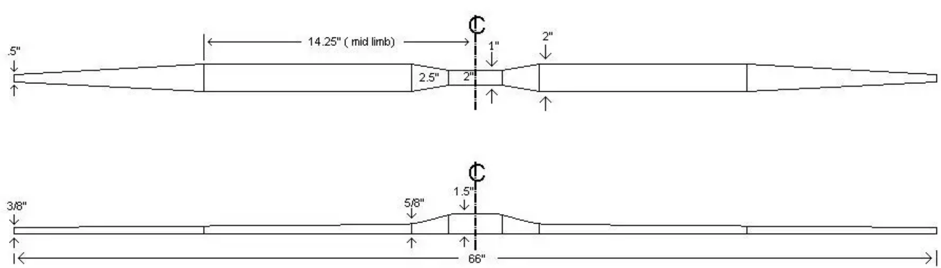
We can integrate this equation once and use the boundary condition

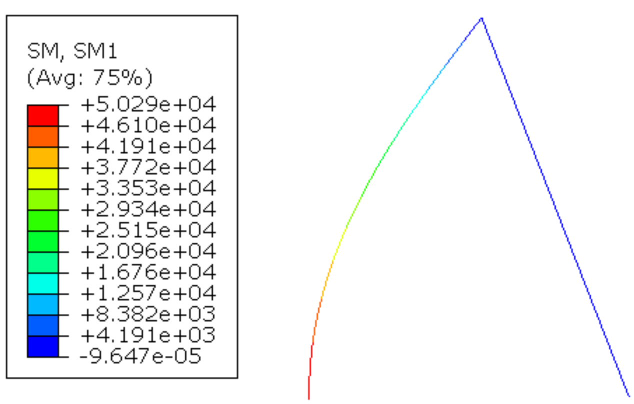
$$\frac{dw}{dx_3} = \frac{b(L - x_3)}{T_0 - c(L - x_3)}$$

We can integrate this again

$$w = \frac{b}{c}\left\{ -x_3 + \frac{T_0}{c}\log\frac{T_0 - c(L - x_3)}{T_0 - cL} \right\}$$

**[5 POINTS]**



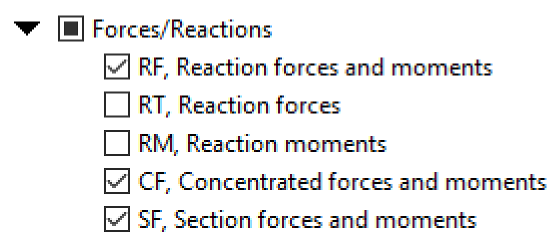
**5.** In this problem you will explore the beam elements available in ABAQUS by setting up a simple finite element simulation to predict the static force-v-draw curve for an English longbow. Dimensions of a simple home-made bow (from this reference) are shown in the figure.

5.1 Because of symmetry, we only need to model half the bow. Create a 3D deformable part with wire base feature, with one end at the origin and the other at y= 830mm (the undeformed bow is straight; it gets bent when the string is attached). Note that this procedure will compute *half* the total draw force (there will be an additional force from the bottom half of the bow that has not been modeled).

5.2 Use a Young's modulus for Yew wood of 6.5 GPa (convert to $N/mm^2$ since lengths are in mm), and make the cross-section a 40mmx12mm rectangle (it is possible to create tapered cross-sections in ABAQUS/CAE but you have to define the taper of every element in the beam separately, which is extremely tedious to do by hand and best done with a python script). Don't forget to assign a beam section orientation – make sure the normal to the flat side of the bow is parallel to the x direction.

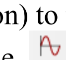
5.3 Create a second part for the bow string. Make one end of the string coincident with the tip of the bow, and make the string 40mm shorter than the bow. Give the bow string a modulus of 110 GPa and a circular cross section with radius 0.72mm.

5.4 Create two steps, both with 1 sec duration – these will be used to string the bow, and then draw it. Make sure NLGEOM is selected for both steps. Make the initial and maximum increment size 0.1s for both steps. In the step module for the first step, open the Field Output Request manager and add requests to plot the section forces/moments. Also, (i) create a new Set (use Tools > Set > Create… name the set drawpoint, Select Geometry, and select the bottom end of the bowstring); then (ii) create a history output request to request displacements/velocity/acceleration and forces/reactions for this set (this will be used to plot the force-v-draw curve).




5.5 To tie the bowstring to the bow, select Constraint > Create …, select a tie constraint; in the menu below the viewport select Node Region, then select the end points of the string and bow as the master and slave (it doesn't matter which is which). If you need to, you can right click a part under the 'instance' branch of the model tree to hide it. In the 'edit constraint' menu uncheck the 'Tie rotational DOFs if applicable' option

5.6 In the boundary condition module, create a BC for the 'initial' step that makes all the degrees of freedom (U and UR) to zero for the node on the bow that lies at the origin. This BC will be applied throughout the analysis by default.

5.7 We need to define boundary conditions for the two load steps. The first step will be used to string the bow. To do this we need to apply a force to bend the bow, and pull the string down so its bottom end lies at y=0. Apply a horizontal force (acting in the positive x direction) to the bow at its tip with magnitude 20 N (if your bow is vertical, the force will be horizontal). Click the  button next to the drop down menu to define an amplitude for this force; check the 'tabular' radio button, and enter a table that starts at 1 at time t=0 and ends at 0 at time t=1. Then select Amp-1 in the dropdown menu. This will bend the bow so it can be pulled down by the bowstring, but will remove the force by the end of the step. Next, create a BC that will apply a displacement to the free end of the bowstring that will pull it down to be level with the base of the bow (this will be U2=-40mm). A standard 'Ramp' amplitude will work. Finally open both the Load and BC managers and deactivate the two BCs created for step 1 in step 2.

5.8 To draw the bow, (i) create a BC that will fix U2 at the bottom of the bowstring to remain constant in the second step. To do this open the BC manager, and make sure the BC you defined at the end of the string in Step 1 is deactivated in Step 2. Then create a new BC, select the bottom end of the bowstring, select 'Fixed at Current Position' in the Edit BC window, and check the box for U2. Next , create a second BC that will apply a constant horizontal velocity (V1) of 200 mm/sec to the bottom end of the bowstring. This will draw the bow by 200mm by the end of the step.

5.9 In the mesh module seed the bow and string with 10mm/20mm mesh size, respectively, and mesh both parts with the default element type.

5.10 Submit and run the job. If all works correctly you should see a deformed shape and bending moment distribution that looks something like the picture on the previous page.

5.11 Create and hand in a plot of the force-v-draw curve for the bow. Work out how to correct the draw force to double the reaction force. For a rough comparison, you can find measured force-v-draw curves for a few bows on this page – but those bows are made from different materials and have a different cross-section to the one analyzed here (which seems to be a rather light bow).

**[10 POINTS]**