



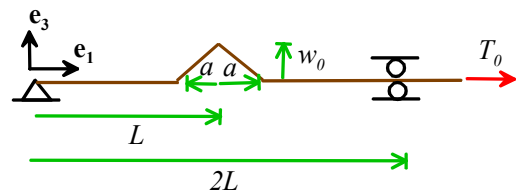
School of Engineering
Brown University

EN1750: Advanced Mechanics of Solids

Homework 8: Dynamics.

Due Friday Nov 30, 2018

1. The figure shows a stretched string with an initial triangular displacement near its center. Assume the string has a wave speed c . Sketch the shape of the string at the following times (you don't need to solve all the equation from scratch, just use your physical understanding of wave propagation and reflection to work out what you expect to see – the animations on the slides shown in class are in the detailed class notes and might be helpful)

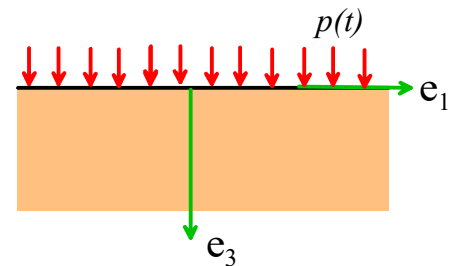


1.1 $t = L / 2c$

1.2 $t = 3L / 2c$

1.3 $t = 7L / 2c$.

2. The surface of an infinite linear elastic half-space with Young's modulus E and Poisson's ratio ν . It is at rest for time $t < 0$, and is subjected to a harmonic pressure on its surface, given by $p(t) = p_0 \sin \omega t$ $t > 0$, with $p=0$ for $t < 0$.



2.1 What are the distributions of stress and velocity in the solid (you can express your answer in terms of $p_0, \omega, t, x_3, \rho, \nu$ and the wave speed c_L . You can just write down the solution by inspection; – use the equation of motion from class for the same problem with a constant force applied at the surface, and use the solution for a string forced at one end to work out what you expect to happen in this problem)?

2.2 Hence, calculate the displacement of the solid at the surface as a function of time.

2.3 Calculate the total work done by the applied pressure in one cycle of loading and hence determine the rate of work done by the pressure. This energy is radiated in kinetic energy away from the surface (ie the power expended by whatever is applying the pressure). Does the rate of work depend on the frequency?

4. [This website](#) lists the vibration modes for a ceramic resonator (an electromechanical circuit element used to make precise electrical oscillator circuits – for further reading you could check [this reference](#) for a huge range of creative static and dynamic applications of piezoelectric materials). The goal of this problem is to calculate a formula for the frequency of the 5th vibration mode (through-thickness vibrations). Assume that the resonator is a rectangular plate with thickness $h \ll L$, and that the displacement in the plate has the form $u_3 = u(x_3)$, with all other components zero.

3.1 Using the general equations for elastic solids (see the lectures on plane waves in infinite solids) show that the equation of motion for u reduces to

$$\frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2} \frac{\partial^2 u}{\partial t^2}$$

and give a formula for c_L

3.2 Assume that the top and bottom surfaces of the plate are stress free. Write down the boundary condition for u at $x_3 = \pm h/2$.

3.3 Consider solutions to the equation of motion of the form $u = \cos(\omega t + \phi) f(x_3)$. Use 3.1 to find an ODE for $f(x_3)$. Find the general solution for f along with the formula relating wave number k to frequency ω (the dispersion relation...)

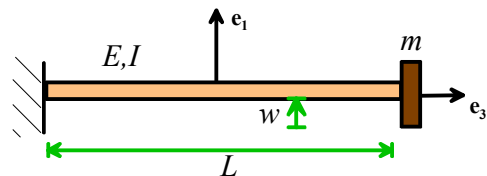
3.4 Show that the boundary conditions can be expressed in matrix form as

$$\begin{bmatrix} \cos kh/2 & \sin kh/2 \\ \cos kh/2 & -\sin kh/2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, find a formula for the resonant wave numbers k and the corresponding resonant frequencies ω . (Set the determinant of the matrix to zero; this will give you an equation for k that will have multiple solutions. Then you can use 3.3 to calculate the natural frequency corresponding to each k).

3.5 Calculate the thickness of a resonator made from PZT with Young's modulus 81 GPa and Poisson's ratio 0.39 and mass density 7320 kg/m³ with a resonant frequency of 10 MHz.

4. Functionalized cantilevers are sometimes used as chemical or biochemical mass sensors (see e.g. [this review article](#)). The basic principle is to detect the change in resonant frequency of the cantilever when a small mass is adsorbed on its tip. The goal of this problem is to provide the necessary relationship between mass and natural frequency. Assume that the beam has modulus E and mass moment of inertia $I_{22} = I_{11} = I$, mass density ρ and cross sectional area A and has a small mass (with negligible mass moment of inertia) attached to its tip (assume that the beam has a symmetric cross section so that $I_{12} = 0$ - you can use the simple beam equation from the example solved in class).



4.1 Draw a free body diagram showing the forces acting on the mass (neglect gravity) and hence show that the transverse force acting on the right end of the cantilever (which is opposite to the force acting on the mass) is related to the displacement at its tip by

$$T_1 = -m \frac{d^2 w}{dt^2}$$

4.2 Write down the differential equation governing flexural vibration of the cantilever, and by considering solutions of the form $w = \cos(\omega t + \phi) f(x_3)$ show that the equation is satisfied by a solution of the form

$$f(x_3) = A \sin kx_3 + B \cos kx_3 + C \sinh kx_3 + D \cosh kx_3$$

Write down the relationship between k, ω , and $\beta = \sqrt{EI / \rho A}$

4.3 Convince yourself that the boundary conditions for the transverse displacement w are:

(i) $w=0$ at $x_3 = 0$

(ii) $\frac{dw}{dx_3} = 0$ at $x_3 = 0$

(iii) Bending moment must be zero at the right hand end so $\frac{d^2 w}{dx_3^2} = 0$ at $x_3 = L$

(iv) The condition from 4.1 requires that $T_1 = -EI \frac{\partial^3 w}{\partial x_3^3} = -m \frac{\partial^2 w}{\partial t^2}$ at $x_3 = L$. Using the answer to 4.2 this can be rewritten as

$$-\frac{\partial^3 w}{\partial x_3^3} + \frac{k^4 L}{\omega^2} \mu \frac{\partial^2 w}{\partial t^2} = 0$$

where $\mu = \frac{m}{A\rho L}$

Using the procedure discussed in class (follow the example for the vibrating free beam, but include the $\cos \omega t$ factor in the solution for w since it is needed in the last boundary condition) show that they can be arranged into the following form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kL) & -\cos(kL) & \sinh(kL) & \cosh(kL) \\ -\cos(kL) + \mu kL \sin(kL) & \sin(kL) + \mu kL \cos(kL) & \cosh(kL) + \mu kL \sinh(kL) & \sinh(kL) + \mu kL \cosh(kL) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4.4 Hence, show that the wave numbers for the vibration modes satisfy

$$\cos(Lk) \cosh(Lk) - \mu kL \cosh(Lk) \sin(Lk) + \mu kL \cos(Lk) \sinh(Lk) + 1 = 0$$

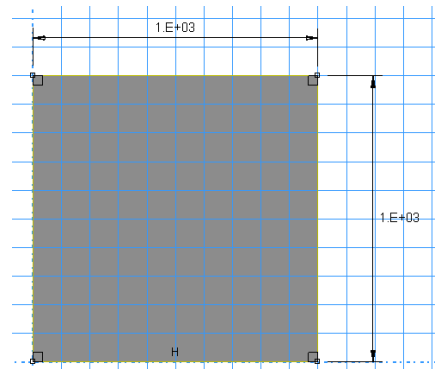
4.5 Calculate the lowest natural frequency of the beam (in terms of β and L) without the mass on its end (i.e. $\mu = 0$). You will need to solve the equation $\cos(Lk) \cosh(Lk) + 1 = 0$ for (Lk) numerically, eg using fsolve in MATLAB. You can google the answer to check it; this is a well known result.

4.6 What would you expect the lowest natural frequency to be in the limit of very large μ ? (this is an engn40/engn310 calculation – give a formula in terms of μ, β, L). You can check your answer using the equation in 4.4.

4.7. Plot a graph of $\omega(\mu) / \omega(0)$, where $\omega(0)$ is the frequency of the cantilever without a tip mass (i.e. the solution to 3.5) as a function of μ for $0 < \mu < 0.2$. You'll need to write a short MATLAB script to do this. Suppose that it is possible to detect a 2% change in frequency, and a typical cantilever has a mass of about 30 ng (nanograms) – your graph should show that the measurement would be able to measure a mass of 200 pg (pictograms) or so. This estimate is comparable to the 400 pg resolution reported in [this paper](#).

5. In this problem you will use explicit dynamic simulations in ABAQUS to study wave propagation near the surface of an elastic solid.

5.1 Create a 2D planar part like the one shown in the figure. The dimension unit is millimeters.



5.2 Create a material with Young's modulus 100 GPa, Poisson's ratio 0.3 and mass density 10000 kg/m³. We will use N for forces and mm for length (so stresses are in N/mm² = MN/m²) – this means 100GPa should be entered as 100000MPa. We need to use a mass density that is consistent with these units. Convince yourself that if we choose to use N for force, mm for length, and milliseconds for time, we must enter density in N milliseconds²/mm⁴, which makes $\rho = 10 \times 10^{-3}$ in our chosen unit system (note that wave speeds in m/s and mm/millisecond are identical). Assign the part a homogeneous section with these properties.

5.3 Create an instance of the part in the usual way.

5.4 Calculate the expected speeds of P, S, and Rayleigh waves in the material (you can assume the Rayleigh wave speed is about 90% of the shear wave speed). Hence, estimate the time (in milliseconds) required for each wave to propagate from one side of the solid to the other. Then create an Explicit Dynamic step with duration roughly twice the time required for the pressure wave to propagate across the solid (enter the time in milliseconds, since that's our chosen time unit). You can turn off NLGEOM for this problem. Use Output->Field Output requests to edit the field output. Change the frequency of output to make sure that at least 200 frames are saved during the analysis (you can change this – you'll get nicer animations with more frequent output, but it will produce huge odb files and slow down the analysis).

5.5 Enter a BC to prevent vertical motion of the base of the block, and apply an instantaneous pressure of 100MPa on the left face.

5.6 Assign an element type of Plane Strain reduced integration elements (CPE4R) from the Explicit element library. Note that the default is plane stress elements so it's important to select the right options in the Element Type menu. Seed the part with a 1.5mm mesh size (if you don't mind waiting a bit longer for the analysis to complete you could try 1mm) and mesh it with quad elements (use a structured meshing algorithm).

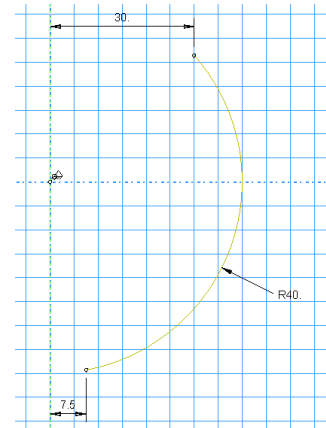
5.7 Submit and run the job. Use Job-> Monitor to track the progress of the analysis – note that since this is an explicit dynamic simulation ABAQUS takes a very large number of very small time-steps.

5.8 You can use the visualization module to watch some fun movies of wave propagation and reflection in the block; you can hopefully figure out what you can see! As a submission for this problem, please plot contours of (i) pressure; (ii) shear stress S12; (iii) velocity V1; and (iv) velocity V2 at a time of around 0.15 milliseconds. Mark on your plots (a) a plane P-wave; (b) a plane S-wave and (c) a Rayleigh wave (these

will show up as changes in stress – for example we expect the pressure to jump across a P-wave but not an S wave; and we will see changes in velocity across any plane wave front). You can double check the wave types by calculating the velocity of each type of wave.

6. In this problem you will calculate the resonant frequencies of an (approximate) wine glass.

6.1 Create a 3D deformable part with shell/revolution base feature. Use something like the sketch shown in the figure (dimensions are in mm), and revolve it through 360 degrees about the centerline to create the glass (fixed BCs will be applied to the small hole at the bottom to represent the stem).



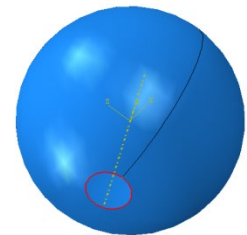
6.2 Create a material with Young's modulus 10GPa, Poisson's ratio 0.2 and mass density 2700 kg/m³. Note that you will have to choose a unit system with lengths in mm – you could use the procedure suggested in the previous problem to do this, but your frequencies will then be reported in cycles/millisecond instead of Hz. You could use forces in N, lengths in mm and time in sec if you prefer but you will need to figure out how to change the density to make this work! Create a homogeneous shell section with 0.8mm thickness and assign it to the part.

6.3 Create an instance of the part

6.4 Create a Linear Perturbation/Frequency step and request 50 or so eigenmodes (you can do fewer if you are in a hurry)

6.5 For BCs, fix all DOF on the bottom edge of the bowl (see the figure)

6.6 Assign a quadratic quad element type with 6DOF per node to the part, seed it with a mesh size 2mm, and mesh it with Sweep algorithm.



6.7 Create/Run the job, and check the mode shapes/frequencies in the visualization module. For comparison, [this video](#) measures a frequency for the 3rd mode (the lowest two are hard to excite with sound) of 317Hz... Hand in a plot showing the mode shapes and frequencies for the first, 3rd, and 15th modes.