



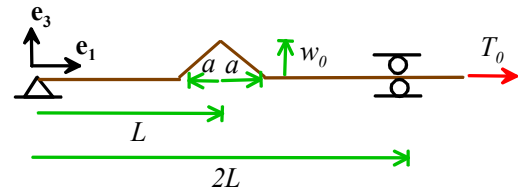
School of Engineering
Brown University

EN1750: Advanced Mechanics of Solids

Homework 8: Dynamics.

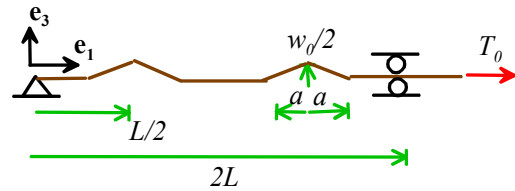
Due Friday Nov 30, 2018

1. The figure shows a stretched string with an initial triangular displacement near its center. Assume the string has a wave speed c . Sketch the shape of the string at the following times (you don't need to solve all the equations from scratch, just use your physical understanding of wave propagation and reflection to work out what you expect to see – the animations in class notes might be helpful)



1.1 $t = L/2c$

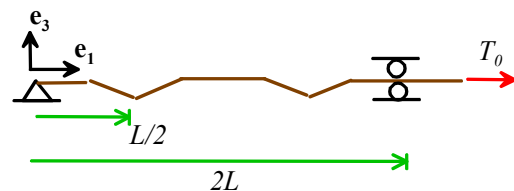
The wave will split into two running in opposite directions, and in time $L/2c$ will run a distance $L/2$.



[2 POINTS]

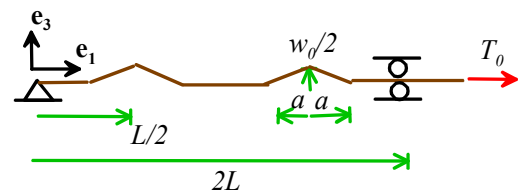
1.2 $t = 3L/2c$

The wave will have reflected off both ends at this point. The wave will flip at each reflection to satisfy the zero displacement boundary condition.



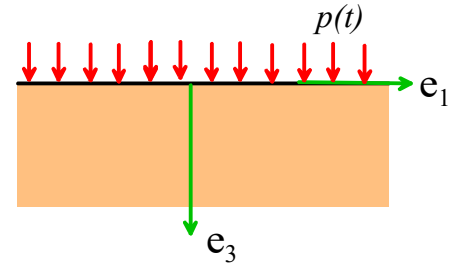
[2 POINTS]

1.3 $t = 7L/2c$. The waves will have traveled a distance $2L$ in between 1.2 and 1.3 – so there will be another reflection off the ends, and we return to the configuration in 1.1.



[2 POINTS]

2. The surface of an infinite linear elastic half-space with Young's modulus E and Poisson's ratio ν . It is at rest for time $t < 0$, and is subjected to a harmonic pressure on its surface, given by $p(t) = p_0 \sin \omega t$ $t > 0$, with $p = 0$ for $t < 0$.



2.1 What are the distributions of stress and velocity in the solid (you can express your answer in terms of $p_0, \omega, t, x_3, \rho, \nu$ and the wave speed c_L)?

From class we know that

$$\sigma_{33} = \begin{cases} -p_0 \sin \omega(t - x_3 / c_L) & t - x_3 / c_L > 0 \\ 0 & t - x_3 / c_L < 0 \end{cases}$$

$$v_3 = \begin{cases} -\frac{p_0 \rho}{c_L} \sin \omega(t - x_3 / c_L) & t - x_3 / c_L > 0 \\ 0 & t - x_3 / c_L < 0 \end{cases}$$

We also know $\epsilon_{11} = \epsilon_{22} = 0$ so the elastic stress-strain relations give

$$\sigma_{11} = \sigma_{22} = \frac{\nu}{(1-\nu)} \sigma_{33}$$

[3 POINTS]

2.2 Hence, calculate the displacement of the solid at the surface as a function of time.

We can just integrate the velocity as a function of time

$$u = \int_0^t -\frac{p_0 \rho}{c_L} \sin(\omega \tau) d\tau = \frac{p_0 \rho}{c_L \omega} (\cos \omega t - 1)$$

[2 POINTS]

2.3 Calculate the total work done by the applied pressure in one cycle of loading and hence determine the rate of work done by the pressure. This energy is radiated in kinetic energy away from the surface (ie the power expended by whatever is applying the pressure). Does the rate of work depend on the frequency?

$$\text{The work done is } W = \int_0^{2\pi/\omega} -p(\tau) v_3(\tau) d\tau = \int_0^{2\pi/\omega} \frac{p_0^2 \rho}{c_L} \sin^2 \omega \tau d\tau = \frac{\pi p_0^2 \rho}{\omega c_L}$$

The power is $W \omega / 2\pi = p_0^2 \rho / c_L$. There is no frequency dependence.

[2 POINTS]

4. [This website](#) lists the vibration modes for a ceramic resonator (an electromechanical circuit element used to make precise electrical oscillator circuits – for further reading you could check [this reference](#) for a huge range of creative static and dynamic applications of piezoelectric materials). The goal of this problem is to calculate a formula for the frequency of the 5th vibration mode (through-thickness vibrations). Assume that the resonator is a rectangular plate with thickness $h \ll L$, and that the displacement in the plate has the form $u_3 = u(x_3)$, with all other components zero.

3.1 Show that the equation of motion for u reduces to

$$\frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2} \frac{\partial^2 u}{\partial t^2}$$

and give a formula for c_L

Follow the derivation of the plane wave solutions in class

$$\sigma_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \varepsilon_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \frac{\partial u}{\partial x_3}$$

The linear momentum balance equation then gives

$$\frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2} \frac{\partial^2 u}{\partial t^2}$$

$$\text{With } c_L = \sqrt{E(1-\nu) / \rho(1+\nu)(1-2\nu)}$$

[3 POINTS]

3.2 Assume that the top and bottom surfaces of the plate are stress free. Write down the boundary condition for u at $x_3 = \pm h / 2$.

The zero stress boundary condition means that the axial strain must be zero at the surfaces:

$$\frac{\partial u}{\partial x_3} = 0 \quad x_3 = \pm h / 2$$

[1 POINT]

3.3 Consider solutions to the equation of motion of the form $u = \cos(\omega t + \phi) f(x_3)$. Use 4.1 to find an ODE for $f(x_3)$. Find the general solution for f along with the formula relating wave number k to frequency ω (the dispersion relation...)

Substitute into the governing equation

$$\frac{\partial^2 f}{\partial x_3^2} \cos(\omega t + \phi) = -\frac{\omega^2}{c_L^2} f \cos(\omega t + \phi) \Rightarrow \frac{\partial^2 f}{\partial x_3^2} + k^2 f = 0$$

This has general solution $f = A \sin kx_3 + B \cos kx_3$ with $k = \omega / c_L$

[2 POINTS]

3.4 Show that the boundary conditions can be expressed in matrix form as

$$\begin{bmatrix} \cos kh/2 & \sin kh/2 \\ \cos kh/2 & -\sin kh/2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, find a formula for the resonant wave numbers k and the corresponding resonant frequencies ω

The boundary condition reduces to $\frac{df}{dx_3} = Ak \cos kx_3 - Bk \sin kx_3$ $x_3 = \pm h/2$

Substituting and writing the equations in matrix form gives the solution stated.

For nontrivial solutions the determinant of the matrix must vanish, which gives $\sin(kh) = 0$. This

has solutions $kh = n\pi \Rightarrow \omega = \frac{n\pi c_L}{h}$

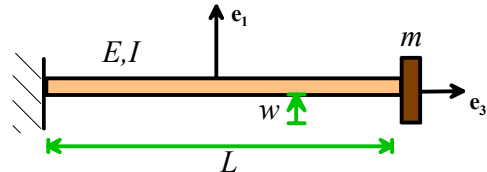
[2 POINTS]

3.5 Calculate the thickness of a resonator made from PZT with Young's modulus 81GPa and Poissons ratio 0.39 and mass density 7320 kg/m³ with a resonant frequency of 10 MHz.

The numbers give a longitudinal wave speed of 4730m/s. The necessary thickness is $\pi c_L / \omega = 0.24\text{mm}$. (don't forget the factor of 2π to calculate the angular frequency...)

[1 POINT]

4. Functionalized cantilevers are sometimes used as chemical or biochemical mass sensors (see e.g. [this review article](#)). The basic principle is to detect the change in resonant frequency of the cantilever when a small mass is adsorbed on its tip. The goal of this problem is to provide the necessary relationship between mass and natural frequency. Assume that the beam has modulus E and mass moment of inertia I , mass density ρ and cross sectional area A and has a small mass (with negligible mass moment of inertia) attached to its tip.



4.1 Draw a free body diagram showing the forces acting on the mass (neglect gravity) and hence show that the transverse force acting on the right end of the cantilever is related to the displacement at its tip by

$$T_2 = -m \frac{d^2 w}{dt^2}$$

This is a simple $F=ma$ problem...

[1 POINT]

4.2 Write down the differential equation governing flexural vibration of the cantilever, and by considering solutions of the form $w = \cos(\omega t + \phi)f(x_3)$ show that the equation is satisfied by a solution of the form

$$f(x_3) = A \sin kx_3 + B \cos kx_3 + C \sinh kx_3 + D \cosh kx_3$$

(you can use exponential solutions of the form $f(x_3) = \sum_{i=1}^4 \exp(\lambda_i x_3)$ if you prefer, where λ_i are the roots of the characteristic equation, but messing with the complex numbers is a bit painful). Write down the relationship between k, ω , and $\beta = \sqrt{EI / \rho A}$

The differential equation is

$$\frac{d^4 w}{dx_3^4} + \frac{1}{\beta^2} \frac{d^2 w}{dt^2} = 0$$

Substituting the solution gives

$$\frac{d^4 f}{dx_3^4} - \frac{\omega^2}{\beta^2} f = 0$$

The given solution for f satisfies the equation with

$$k^4 = \frac{\omega^2}{\beta^2}$$

[2 POINTS]

4.3 Write down the boundary conditions for the transverse displacement w , and show that they can be arranged into the following form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kL) & -\cos(kL) & \sinh(kL) & \cosh(kL) \\ -\cos(kL) + \mu kL \sin(kL) & \sin(kL) + \mu kL \cos(kL) & \cosh(kL) + \mu kL \sinh(kL) & \sinh(kL) + \mu kL \cosh(kL) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mu = \frac{m}{A\rho L}$

The boundary conditions are

1. Zero deflection at $x_3 = 0$, which requires $B + D = 0$
2. Zero slope at $x_3 = 0$, which requires $A + C = 0$
3. Zero bending moment at $x_3 = L$, i.e. $d^2 w / dx_3^2 = 0$

This gives $(-A \sin kL - B \cos kL + C \sinh kL + D \cosh kL)$

4. The transverse force must satisfy the expression from 3.1, which requires

$$-EI \frac{d^3 w}{dx_3^3} = -m \frac{d^2 w}{dt^2}$$

$$\Rightarrow -EI k^3 (-A \cos kL + B \sin kL + C \cosh kL + D \sinh kL) = m \omega^2 (A \sin kL + B \cos kL + C \sinh kL + D \cosh kL)$$

Using the dispersion relation gives

$$-Elk^3(-A \cos kL + B \sin kL + C \cosh kL + D \sinh kL) = mk^4 \frac{EI}{\rho A} (A \sin kL + B \cos kL + C \sinh kL + D \cosh kL)$$

This can be re-written as

$$A(-\cos kL + \mu kL \sin kL) + B(\sin kL + \mu kL \cos kL) + C(\cosh kL + \mu kL \sinh kL) + D(\sinh kL + \mu kL \cosh kL) = 0$$

Collecting all four boundary conditions into matrix form gives

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kL) & -\cos(kL) & \sinh(kL) & \cosh(kL) \\ -\cos(kL) + \mu kL \sin(kL) & \sin(kL) + \mu kL \cos(kL) & \cosh(kL) + \mu kL \sinh(kL) & \sinh(kL) + \mu kL \cosh(kL) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Alternatively we can have MATLAB do the heavy lifting

```
clear all
syms k L mu x3 omega t
w = cos(omega*t)*[sin(k*x3),cos(k*x3),sinh(k*x3),cosh(k*x3)];
N = [subs(w,x3,0);...
     subs(diff(w,x3),x3,0);...
     subs(diff(w,x3,2),x3,L);...
     simplify(subs( (-k^4*L*mu/omega^2*diff(w,t,2)+diff(w,x3,3)),x3,L) )]
```

[5 POINTS]

4.4 Hence, show that the wave numbers for the vibration modes satisfy

$$\cos(Lk)\cosh(Lk) - \mu kL \cosh(Lk)\sin(Lk) + \mu kL \cos(Lk)\sinh(Lk) + 1 = 0$$

Use Matlab...

```
clear all
syms k L mu K real
K = [0,1,0,1;...
     1,0,1,0;...
     -sin(k*L),-cos(k*L),sinh(k*L),cosh(k*L);...
     -cos(k*L)+mu*sin(k*L),sin(k*L)+mu*cos(k*L),cosh(k*L)+mu*sinh(k*L),sinh(k*L)+mu*cosh(k*L)]
simplify(det(K))
```

This gives the answer stated

[2 POINTS]

4.5 Calculate the lowest natural frequency of the beam (in terms of β and L) without the mass on its end (i.e. $\mu = 0$). You will need to solve the equation $\cos(Lk)\cosh(Lk) + 1 = 0$ numerically, eg using fsolve in MATLAB. You can google the answer to check it; this is a well known result.

Solving the equation gives $kL = 1.875$, the dispersion relation gives

$$\omega = k^2 \beta = \frac{(1.875)^2}{L^2} \beta$$

[2 POINTS]

4.6 What would you expect the lowest natural frequency to be in the limit of very large μ ? (this is an engn40/engn310 calculation – give a formula in terms of μ, β, L).

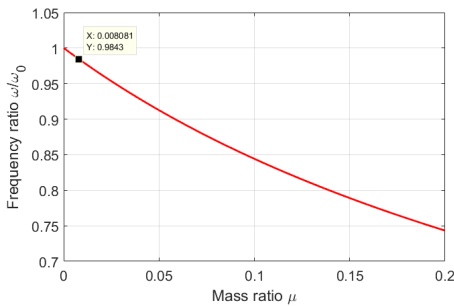
In this limit we can idealize the system as a simple spring mass system; the stiffness of the spring is $\frac{3EI}{L^3}$ so

the natural frequency is $\omega = \sqrt{\frac{3EI}{L^3 m}} = \sqrt{\frac{3}{\mu}} \frac{\beta}{L^2}$. This also tells us that $kL = \left(\frac{3}{\mu}\right)^{1/4}$, which gives us a way to check our formula for the natural frequency (it works!). But the question did not ask for this.

[3 POINTS]

4.7. Plot a graph of $\omega(\mu) / \omega(0)$, where $\omega(0)$ is the frequency of the cantilever without a tip mass (i.e. the solution to 3.5) as a function of $0 < \mu < 0.2$. You'll need to write a short MATLAB script to do this. Suppose that it is possible to detect a 2% change in frequency, and a typical cantilever has a mass of about 30 ng (nanograms) – your graph should show that the measurement would be able to measure a mass of 200 pg (pictograms) or so. This estimate is comparable to the 400 pg resolution reported in [this paper](#).

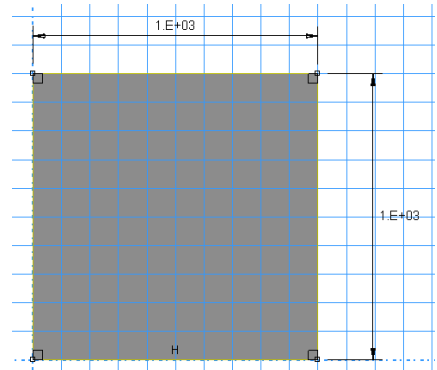
The graph is plotted below.



[3 POINTS]

5. In this problem you will use explicit dynamic simulations in ABAQUS to study wave propagation near the surface of an elastic solid.

5.1 Create a 2D planar part like the one shown in the figure. The dimension unit is millimeters.



5.2 Create a material with Youngs modulus 100 GPa, Poissons ratio 0.3 and mass density 10000 kg/m³. We will use N for forces and mm for length (so stresses are in N/mm² = MN/m²) – this means 100GPa should be entered as 100000MPa. We need to use a mass density that is consistent with these units. Convince yourself that if we choose to use N for force, mm for length, and milliseconds for time, we must

enter density in N milliseconds²/mm⁴, which makes $\rho = 10 \times 10^{-3}$ in our chosen unit system (note that wave speeds in m/s and mm/millisecond are identical). Assign the part a homogeneous section with these properties.

5.3 Create an instance of the part in the usual way.

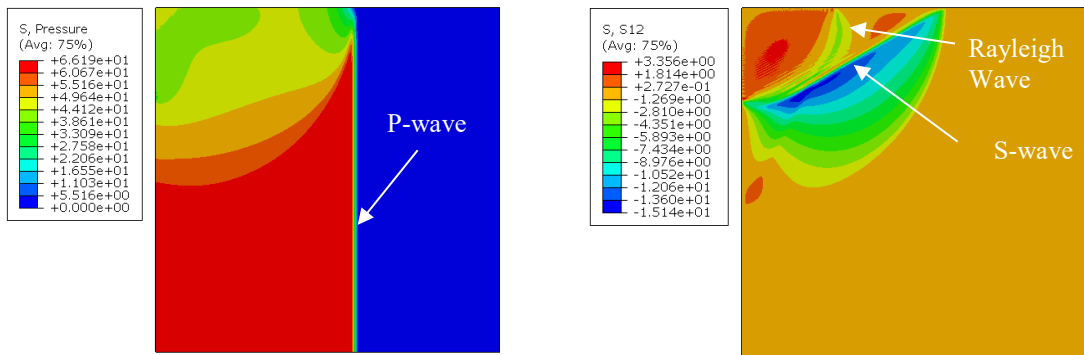
5.4 Calculate the expected speeds of P, S, and Rayleigh waves in the material. Hence, estimate the time (in milliseconds) required for each wave to propagate from one side of the solid to the other. Then create an Explicit Dynamic step with duration roughly twice the time required for the pressure wave to propagate across the solid (enter the time in milliseconds, since that's our chosen time unit.). You can turn off NLGEOM for this problem. Use Output->Field Output requests to edit the field output. Change the frequency of output to make sure that at least 200 frames are saved during the analysis (you can change this – you'll get nicer animations with more frequent output, but it will produce huge odb files and slow down the analysis).

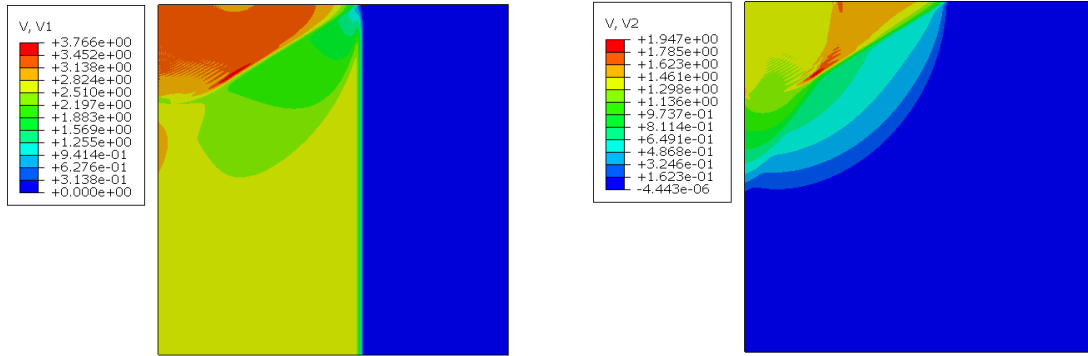
5.5 Enter a BC to prevent vertical motion of the base of the block, and apply an instantaneous pressure of 100MPa on the left face.

5.6 Assign an element type of Plane Strain reduced integration elements (CPE4R) from the Explicit element library. Note that the default is plane stress elements so it's important to select the right options in the Element Type menu. Seed the part with a 1.5mm mesh size (if you don't mind waiting a bit longer for the analysis to complete you could try 1mm) and mesh it with quad elements (use a structured meshing algorithm).

5.7 Submit and run the job. Use Job-> Monitor to track the progress of the analysis – note that since this is an explicit dynamic simulation ABAQUS takes a very large number of very small time-steps.

5.8 You can use the visualization module to watch some fun movies of wave propagation and reflection in the block; you can hopefully figure out what you can see! As a submission for this problem, please plot contours of (i) pressure; (ii) shear stress S12; (iii) velocity V1; and (iv) velocity V2 at a time of around 0.15 milliseconds. Mark on your plots (a) a plane P-wave; (b) a plane S-wave and (c) a Rayleigh wave (these will show up as changes in stress – for example we expect the pressure to jump across a P-wave but not an S wave; and we will see changes in velocity across any plane wave front). You can double check the wave types by calculating the velocity of each type of wave.

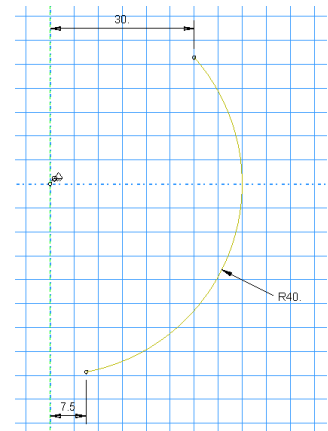




[10 POINTS]

6. In this problem you will calculate the resonant frequencies of an (approximate) wine glass.

6.1 Create a 3D deformable part with shell/revolution base feature. Use something like the sketch shown in the figure (dimensions are in mm), and revolve it through 360 degrees about the centerline to create the glass (fixed BCs will be applied to the small hole at the bottom to represent the stem).



6.2 Create a material with Young's modulus 10GPa, Poisson's ratio 0.2 and mass density 2700 kg/m³. Note that you will have to choose a unit system with lengths in mm – you could use the procedure suggested in the previous problem to do this, but your frequencies will then be reported in cycles/millisecond instead of Hz. You could use forces in N, lengths in mm and time in sec if you prefer but you will need to figure out how to change the density to make this work! Create a homogeneous shell section with 0.8mm thickness and assign it to the part.

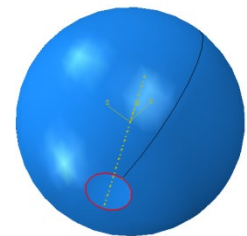
6.3 Create an instance of the part

6.4 Create a Linear Perturbation/Frequency step and request 50 or so eigenmodes (you can do fewer if you are in a hurry)

6.5 For BCs, fix all DOF on the bottom edge of the bowl (see the figure)

6.6 Assign a quadratic quad element type with 6DOF per node to the part, seed it with a mesh size 2mm, and mesh it with Sweep algorithm.

6.7 Create/Run the job, and check the mode shapes/frequencies in the visualization module. For comparison, [this video](#) measures a frequency for the 3rd mode (the lowest two are hard to excite with sound) of 317Hz... Hand in a plot showing the mode shapes and frequencies for the first, 3rd, and 15th modes.



[10 POINTS]