School of Engineering
Brown University

# EN1750: Advanced Mechanics of Solids

## Homework 4: Internal forces and equations of motion.

Due Friday Oct 11, 2019

**1.** For the stress tensor

$$\boldsymbol{\sigma}=\begin{bmatrix}-400 & 300 & 0\\ 300 & 400 & 0\\ 0 & 0 & 600\end{bmatrix}$$

please calculate

1.1 The hydrostatic stress

$$\sigma_h = \text{trace}(\boldsymbol{\sigma})/3 = 600/3 = 200$$

**[1 POINT]**

1.2 The principal stresses and their directions

$$\det(\boldsymbol{\sigma}-\lambda\mathbf{I})=0$$
$$\Rightarrow (600-\lambda)\left(-(400-\lambda)(400+\lambda)-300^2\right)=0$$
$$\Rightarrow \lambda^2-250000=0 \Rightarrow \lambda=\pm 500$$

$$\text{So } [\sigma_1,\sigma_2,\sigma_3]=[600,500,-500]$$

The principal stress directions are the null vectors of

$$\begin{bmatrix}-400-\lambda & 300 & 0\\ 300 & 400-\lambda & 0\\ 0 & 0 & 600-\lambda\end{bmatrix}$$

$$\mathbf{m}^1=[0,0,1] \qquad \mathbf{m}^2=[3,9,0]/\sqrt{90} \qquad \mathbf{m}^3=[-9,3,0]/\sqrt{90}$$

**[3 POINTS]**

1.3 The tractions acting on a plane with normal $\mathbf{n}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$

$$\mathbf{n}\cdot\boldsymbol{\sigma}=\frac{1}{\sqrt{3}}[1,1,1]\begin{bmatrix}-400 & 300 & 0\\ 300 & 400 & 0\\ 0 & 0 & 600\end{bmatrix}=[-100,700,600]/\sqrt{3}$$

**[1 POINT]**

**2.** The "Christensen" failure criterion is used to predict the strength of a material that fails by either yielding under compressive loads, or brittle fracture under tensile loads. It predicts that a material will fail if the principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ satisfy

$$\left(\frac{1}{T}-\frac{1}{C}\right)(\sigma_1+\sigma_2+\sigma_3)+\frac{1}{2TC}\left\{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right\}\geq 1$$

Where $T$ and $C$ are the magnitudes of stress that will cause failure in uniaxial tension and compression, respectively.

This paper reports tensile and compressive failure stresses $T = 6.9MPa$ and $C = 94MPa$ for Carrara Marble.

2.1What is the failure stress of marble subjected to hydrostatic tension?

In hydrostatic compression $\sigma_1 = \sigma_2 = \sigma_3 = p$ which gives

$$\left(\frac{1}{6.9}-\frac{1}{94}\right)(3p)=1 \Rightarrow p = 2.49MPa$$

**[1 POINT]**

2.2 What is the failure stress of the marble subjected to shear (eg in a torsion test)?

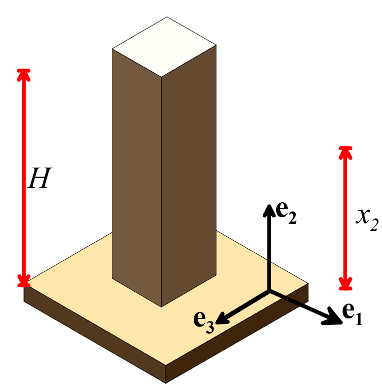
In shear, the principal stresses are the eigenvalues of $\begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$, i.e. $\sigma_1 = \tau, \qquad \sigma_2 = -\tau$

This gives

$$\frac{1}{2\times 6.9\times 94}\left\{2\tau^2\right\}=1 \Rightarrow \tau = 25.5MPa$$

**[2 POINTS]**

**3.** A prismatic concrete column of mass density $\rho$ supports its own weight, as shown below. (Assume that the solid is subjected to a uniform gravitational body force of magnitude $g$ per unit mass).




3.1 Show that the stress distribution

$$\sigma_{22} = -\rho g(H - x_2)$$

satisfies the equations of static equilibrium

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0$$

and also satisfies the boundary conditions $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{0}$ on all free boundaries.

The equilibrium equations in the 1 and 3 direction are trivially satisfied. In the 2 direction the equilibrium equation is

$$\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_3}+b_3=0\Rightarrow\rho g-\rho g=0$$

The boundary conditions on the top surface are $\mathbf{e}_2\cdot\boldsymbol{\sigma}=\mathbf{0}\Rightarrow\sigma_{21}=\sigma_{22}=\sigma_{23}=0$ , which is satisfied.

The boundary conditions on sides with normal in the $\mathbf{e}_3$ direction is $\mathbf{e}_3\cdot\boldsymbol{\sigma}=\mathbf{0}\Rightarrow\sigma_{31}=\sigma_{32}=\sigma_{33}=0$

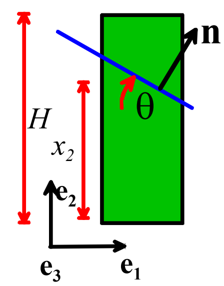
The boundary conditions on sides with normal in the $\mathbf{e}_1$ direction is $\mathbf{e}_1\cdot\boldsymbol{\sigma}=\mathbf{0}\Rightarrow\sigma_{11}=\sigma_{12}=\sigma_{13}=0$

**[3 POINTS]**

3.2 Find a formula for the traction vector acting on a plane with normal $\mathbf{n}=\sin\theta\,\mathbf{e}_1+\cos\theta\,\mathbf{e}_2$ at a height $x_2$

$$\mathbf{T}=\mathbf{n}\cdot\boldsymbol{\sigma}=\begin{bmatrix}\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \sigma\end{bmatrix}=-\rho g(H-x_2)\cos\theta\mathbf{e}_2$$

**[1 POINT]**




3.3 Find the normal and tangential tractions acting on the plane with normal $\mathbf{n}=\sin\theta\,\mathbf{e}_1+\cos\theta\,\mathbf{e}_2$ at a height $x_2$

$$T_n=\mathbf{n}\cdot\mathbf{T}=-\rho g(H-x_2)\cos^2\theta$$

and show also that the tangential component of traction acting on the plane is

$$\begin{aligned}\mathbf{T}_t&=\mathbf{T}-T_n\mathbf{n}=-\rho g(H-x_2)\cos\theta\mathbf{e}_2+\rho g(H-x_2)\cos^2\theta(\sin\theta\mathbf{e}_1+\cos\theta\mathbf{e}_2)\\&=\rho g(H-x_2)\sin\theta\cos\theta(\cos\theta\mathbf{e}_1-\sin\theta\mathbf{e}_2)\end{aligned}$$

**[2 POINTS]**

3.4 Suppose that the concrete contains a large number of randomly oriented microcracks. A crack which lies at an angle $\theta$ to the horizontal will propagate if

$$|\mathbf{T}_t|+\mu T_n>\tau$$

where $\mu$ is the friction coefficient between the faces of the crack and $\tau$ is a critical shear stress that is related to the size of the microcracks and the fracture toughness of the concrete.

Assume that $\mu=1$. Find the orientation of the microcrack that is most likely to propagate. Hence, find an expression for the maximum possible height of the column.

We have that

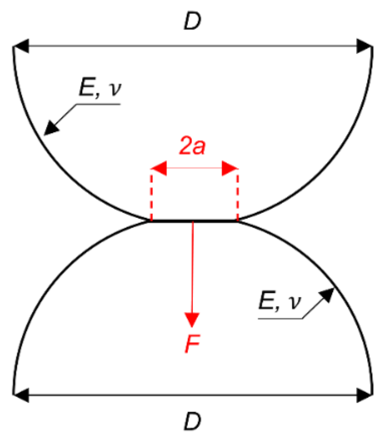
$$|\mathbf{T}_t|+\mu T_n=\rho g(H-x_2)\sin\theta\cos\theta-\mu\rho g(H-x_2)\cos^2\theta=\rho g(H-x_2)\cos\theta(\sin\theta-\mu\cos\theta)$$

To find the critical crack orientation we need to maximize $\cos\theta(\sin\theta-\cos\theta)$ (assuming $\mu=1$ ), which gives $\theta=3\pi/8$ ; substituting back gives $0.2071\rho g(H-x_2)=\tau$ at failure; the critical location is at $x_2=0$ and therefore

$$H = 4.83\tau / (\rho g)$$

**[2 POINTS]**

**4.** The contact pressure distribution acting between two axially symmetric elastic solids has the form




$$p(x,y) = p_0\sqrt{1-r^2/a^2} \qquad r = \sqrt{x^2+y^2}$$

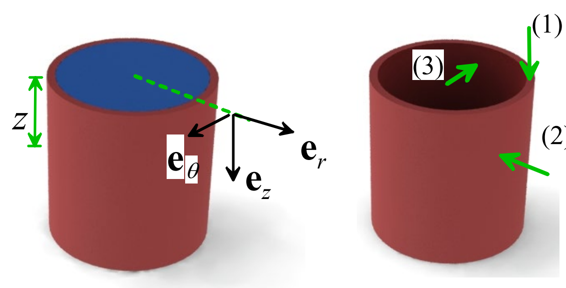
Where $a$ is the radius of the contact patch, and $p_0$ is the maximum contact pressure. Find an expression for the force acting on the contacting solids in terms of $p_0, a$ .

The resultant force is

$$F = \int_0^a p(r)2\pi r dr = \frac{2\pi}{3}a^2 p_0$$

**[2 POINTS]**

**5.** The figure shows a tumbler with mass density $\rho_T$ filled with fluid with mass density $\rho$ . Using the cylindrical-polar basis shown, write down the boundary conditions for stress components on the three surfaces marked (1), (2), (3) in the figure.




(1) (Top surface, $z=0, \quad a<r<a+t$ )

$\sigma_{zz}=0, \ \sigma_{rz}=0 \ \ \sigma_{z\theta}=0$

(2) (Outer surface $r=a+t$)

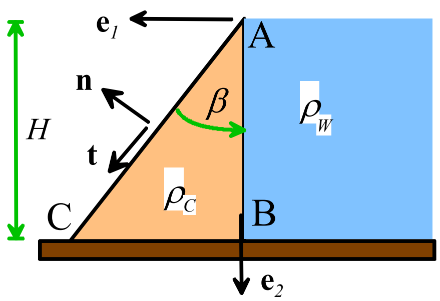
$\sigma_{rr}=0, \ \sigma_{rz}=0 \ \ \sigma_{r\theta}=0$

(3) (Inner surface $r=a$)

$\sigma_{rr}=-\rho g z, \ \sigma_{rz}=0 \ \ \sigma_{r\theta}=0$

**[3 POINTS]**

**6.** The figure shows a simple design for a dam. The wedge-shaped dam is made from a material with weight density $\rho_C$ . It is loaded on its vertical face by fluid pressure $p = -\rho_W x_2$ , where $\rho_W$ is the weight density of the fluid.




6.1 Write down formulas for unit vectors **t, n** tangent and normal to the back face of the dam, in terms of $\beta$ (you will need this information for part 7.2 and 7.3)

$$\mathbf{t} = \sin\beta \mathbf{e}_1 + \cos\beta \mathbf{e}_2$$
$$\mathbf{n} = -\sin\beta \mathbf{e}_2 + \cos\beta \mathbf{e}_1$$

**[2 POINTS]**

6.2 Write down the boundary conditions on the two faces AB, AC of the dam, in terms of the 2D stress components $\sigma_{11}, \sigma_{22}, \sigma_{12}$ .

On face AB, the condition $\mathbf{n}\cdot\boldsymbol{\sigma} = \rho_W x_2 \mathbf{e}_1$ gives $\left[-1, 0\right]\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}-\sigma_{11}\\ -\sigma_{12}\end{bmatrix} = \begin{bmatrix}\rho_w x_2\\ 0\end{bmatrix}$

On face AC, the condition $\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{0}$ gives $\left[\cos\beta, -\sin\beta\right]\begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{bmatrix} = \begin{bmatrix}\sigma_{11}\cos\beta - \sigma_{12}\sin\beta\\ \sigma_{12}\cos\beta - \sigma_{22}\sin\beta\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

**[2 POINTS]**

6.2 Consider the stress state

$$\sigma_{11} = -\rho_W x_2$$
$$\sigma_{22} = \rho_C\left(x_1\cot(\beta) - x_2\right) - \rho_W\cot^2\beta(2x_1\cot(\beta) - x_2)$$
$$\sigma_{12} = -\rho_W x_1\cot^2\beta$$

Show that the stress state satisfies equilibrium in the dam (inside the wedge) and also satisfies the boundary conditions on the faces AB and AC. Note that on face AC $x_1 = x_2\tan\beta$ .

The equilibrium equations are

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} = 0$$
$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \rho_C = -\rho_W\cot^2\beta - \rho_C + \rho_W\cot^2\beta + \rho_C = 0$$

**[2 POINTS]**

The boundary condition on face AB is $\sigma_{11} = -\rho_W x_2 \qquad \sigma_{12} = 0 \qquad x_1 = 0$ . This is clearly satisfied.

**[2 POINTS]**

The boundary condition on face AC is

$$t_1 = \sigma_{11}\cos\beta - \sigma_{12}\sin\beta = 0$$
$$t_2 = \sigma_{12}\cos\beta - \sigma_{22}\sin\beta = 0$$

on $x_1 = s\sin\beta, \qquad x_2 = s\cos\beta$ Substituting the given formulas

$$t_1 = -\rho_W s\cos^2\beta + \rho_W s\sin^2\beta\cot^2\beta = 0$$
$$t_2 = -\rho_W s\sin\beta\cos\beta\cot^2\beta - \rho_C(s\sin\beta\cot\beta - s\cos\beta) + \rho_W\cot^2\beta\sin\beta(2s\sin\beta\cot\beta - s\cos\beta) = 0$$

**[2 POINTS]**

6.3 Suppose that the dam is made from concrete, which cannot safely withstand tensile stress. Assuming that the maximum principal tensile stress occurs at B, find a formula for the minimum allowable value for the angle $\beta$ , in terms of $\rho_c, \rho_W$

The condition

$$\sigma_{22} < 0 \quad x_1 = 0$$
$$\Rightarrow \rho_C\left(-x_2\right) - \rho_W\cot^2\beta(-x_2) < 0 \Rightarrow \cot^2\beta < \rho_C / \rho_W$$

**[2 POINTS]**

6.4 Assume that the concrete fails by crushing if the minimum principal stress reaches $\sigma_3 = -\sigma_c$ . Assuming that the minimum principal stress occurs at point C, and that $\beta$ has the minimum value calculated in 7.3, find an expression for the maximum height of the dam, in terms of $\rho_C, \rho_W$ . You may find the trig identity $1 + \tan^2\beta = 1/\cos^2\beta$ helpful.

Since the rear face of the dam is free of traction, the direction of the nonzero principal stress must be parallel to **t**. Therefore

$$\sigma_3 = \mathbf{t}\cdot\boldsymbol{\sigma}\mathbf{t} = \left[\sin\beta \quad \cos\beta\right]\begin{bmatrix}\sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix}\sin\beta \\ \cos\beta\end{bmatrix} = \sigma_{11}\sin^2\beta + \sigma_{22}\cos^2\beta + 2\sigma_{12}\sin\beta\cos\beta$$

Note that at C $x_1 = H\tan\beta \qquad x_2 = H$ so

$$\sigma_{11} = -\rho_W H \qquad \sigma_{22} = -\rho_W H\cot^2\beta \qquad \sigma_{12} = -\rho_W\cot\beta$$

This gives

$$\sigma_3 = -\rho_W H\left(\sin^2\beta + \cos^2\beta\cot^2\beta + 2\sin\beta\cos\beta\cot\beta\right)$$

$$= -\rho_W H\left(1 + \cos^2\beta + \cos^2\beta\cot^2\beta\right) = -\frac{\rho_W H}{\sin^2\beta}\left(\sin\beta^2 + \cos^2\beta\sin^2\beta + \cos^4\beta\right) = -\frac{\rho_W H}{\sin^2\beta}$$

From 7.3 we know that

$$\tan^2\beta = \rho_W/\rho_C \Rightarrow \frac{1}{\cos^2\beta} = 1 + \rho_W/\rho_C \Rightarrow \sin^2\beta = 1 - \frac{1}{1+\rho_W/\rho_C} = \frac{\rho_W}{\rho_W+\rho_C}$$

So $\sigma_3 = -H(\rho_W + \rho_C)$ , and therefore $H < \sigma_c/(\rho_W + \rho_C)$

**[4 POINTS]**

**7. Importing CAD data into ABAQUS**. Work through the tutorial on this topic. Then, use Solidworks to create a simple part, import it into ABAQUS (try both a step or iges import and a .inp file import. Its best to avoid the STL file import), and run a simulation. Keep everything simple, or you will run into problems. Try to make your part differ from those submitted by your classmates!

Please upload your solidworks model and cae file to canvas as a solution to this problem.

**[10 POINTS]**