1. An overhead transmission cable with mass density ρ and tensile strength σ_{UTS} has a uniform cross-sectional area A. IEEE standard 524 recommends that the tension in the cable should not exceed 10% of the strength of the cable.
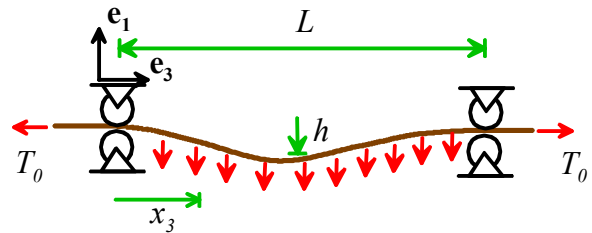
1.1 Find a formula for the sag h at mid-span of the cable, in terms of the cable length L, ρ and σ_{UTS} (you can assume small deflections, and use the solution for the cable deflection calculated in class).

The strength of the cable is $A\sigma_{UTS}$; the weight per unit length is $\rho g A$. From class, the deflected shape is

$u_2 = \dfrac{q}{2T_3} x_3(L - x_3)$. At mid-span, therefore

$$h = \frac{\rho g}{2 \times 0.1 \times \sigma_{UTS}} \frac{L^2}{4}$$

[2 POINTS]

1.2 Find the maximum length of a <u>high conductivity copper</u> cable with mass density 9 g/cm^3 and tensile strength 170 MPa if h/L is required to be less than 0.1

Substituting numbers gives $L < \dfrac{0.1 \times 0.8\sigma_{UTS}}{\rho g} = \dfrac{0.08 \times 170 \times 10^6}{9.8 \times 9 \times 1000} = 154m$

[2 POINTS]

(For interest – this <u>accident report</u> is an excellent example of a well written engineering report and includes some relevant solid mechanics calculations related to cable deflections)

2. A cable with weight per unit length p is stretched by tension T_0 and subjected to a transverse force Q at mid-span. Calculate the deflection h of the cable at mid-span. What value of Q (in term of p,L) will make the displacement at mid-span zero? You can assume small deflections.

We can consider the section of the cable $0 < x_3 < L/2$. For small deflections the tension in the cable is constant. A free body diagram for a small section of the cable at mid-span shows that

$$\sin\theta \approx \frac{du_1}{dx_3} = \frac{Q}{2T_0} \qquad x_3 = L/2$$
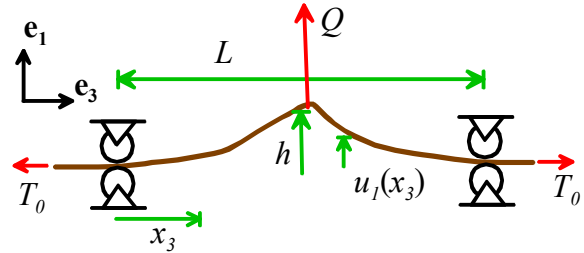
The governing equation for transverse deflections of a cable (from class) is

$$T_0\frac{d^2u_1}{dx_3^2} - p = 0$$

This has solution

$$u_1 = \frac{px_3^2}{2T_0} + Ax_3 + B$$

The transverse deflection is zero at $x_3 = 0$, which shows that $B = 0$. The slope at $x_3 = L/2$ is then

$$\frac{du_1}{dx_3} = \frac{p}{T_0}\frac{L}{2} + A = \frac{Q}{2T_0} \Rightarrow A = \frac{Q - pL}{2T_0}$$

The deflection at mid-span therefore follows as

$$u_1 = \frac{pL^2}{8T_0} + \frac{(Q - pL)}{4T_0}L = \frac{QL}{4T_0} - \frac{pL^2}{8T_0}$$

For zero displacement at the center $Q=pL/2$ – this makes sense, since a support at the center would carry half the weight of the cable.

[5 POINTS]

3. The figure shows the cross-section of a beam.

Find the position of the centroid, and calculate the area moments of inertia of the cross section. Simplify your answer by assuming that $b=a+t$, with $t<<a$ (i.e. take the Taylor expansion to first order in t). You can do the calculation in MATLAB – cut and paste a screenshot into your HW solution if you do this.
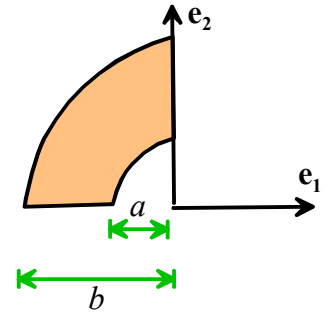
A Matlab solution is shown below.

```
clear all
syms eta theta r rbar a b t A II x1 x2 real
assume(a>0); assume(b>a)
% Compute the area
A = int(int(eta,eta,[a,b]),theta,[pi/2,pi])
x1 = eta*cos(theta); x2 = eta*sin(theta)   ;
r = [x1,x2];
% Find the centroid (neutral line)
rbar = simplify(int(int(r*eta,eta,[a,b]),theta,[pi/2,pi])/
% Calculate the inertia tensor
integrand = [(r(2)-rbar(2))^2, -(r(1)-rbar(1))*(r(2)-rbar(
              -(r(1)-rbar(1))*(r(2)-rbar(2)),(r(1)-rbar(1))
II = simplify(int(int(integrand*eta,eta,[a,b]),theta,[pi/2
II = subs(II,b,a+t);
series(II,t,'Order',1)
```
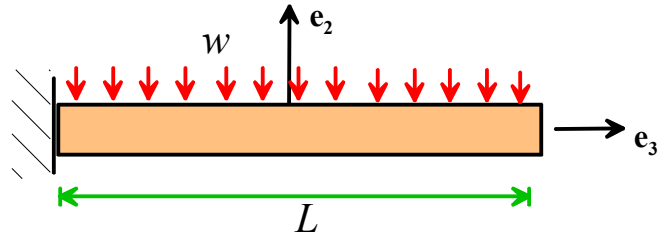
rbar =

$$\left(-\frac{4\left(\frac{a^3}{3}-\frac{b^3}{3}\right)}{\pi\ (a^2-b^2)} \quad \frac{4\left(\frac{a^3}{3}-\frac{b^3}{3}\right)}{\pi\ (a^2-b^2)} \right)$$

ans =

$$\left(\begin{array}{cc} t\left(\frac{\pi\ a^3}{4}-\frac{2\ a^3}{\pi}\right) & t\left(\frac{a^3}{2}-\frac{2\ a^3}{\pi}\right) \\ t\left(\frac{a^3}{2}-\frac{2\ a^3}{\pi}\right) & t\left(\frac{\pi\ a^3}{4}-\frac{2\ a^3}{\pi}\right) \end{array} \right)$$

[5 POINTS]

4. Suppose that a cantilever beam with length L and cross-section analyzed in problem 3 is subjected to a uniform load w acting in the (negative) $\mathbf{e}_2$ direction. Calculate the deflection of the beam at its tip (you can use the Matlab script from class)

The governing equations are

$$E\left(I_{22}\frac{d^4u_1}{dx_3^4} + I_{12}\frac{d^4u_2}{dx_3^4}\right) = 0 \qquad E\left(I_{12}\frac{d^4u_1}{dx_3^4} + I_{11}\frac{d^4u_2}{dx_3^4}\right) = -w$$

The boundary conditions are zero displacement and slope at the left hand end of the beam.

It's easy to solve the equations with MATLAB. As discussed in class, it is easiest to satisfy the boundary conditions at the free end by solving for deflections in a beam with prescribed displacement and rotation at both the ends first, and then calculating the deflection/rotation at the free end that will ensure the moment and shear force vanishes.

```matlab
% Script to calculate deflections of a beam
% You can easily adapt this script to other loads and boundary conditions
syms L a t I11 I12 I22 x3 EE C1 C2 C3 C4 w real
syms u1(x3) u2(x3)
% The second moments of area
I11 = II(1,1);
I12 = -II(1,2); % A positive off diagonal means I12 is a negative number
I22 = II(2,2);
u = [u1;u2];
eq = EE*[I11,I12;I12,I22]*diff(u,x3,4)==[0;-w];
BC1 = subs(u,x3,0)==[0;0];  % Fix u=0 at x3=0
BC2 = subs(diff(u,x3),x3,0)==[0;0]; % Fix du/dx3 = 0 at x3=0
% For some reason MATLAB can't handle the BCs we want, so
% we fix u and du/dx3 to some arbitrary values at x3=L,
% and then solve for these values later
BC3 = subs(u,x3,L)==[C1;C2];
BC4 = subs(diff(u,x3),x3,L)==[C3;C4];
[u1,u2] = dsolve(eq,BC1,BC2,BC3,BC4);
curv = diff([-u2;u1],x3,2);   % Curvature vector
M = EE*II*curv;  % Moment
T = diff([-M(2);M(1)],x3);    % Shear force
BC1 = subs(M,x3,L) ==[0;0];    % Set zero moment at x3=L
BC2 = subs(T,x3,L)==[0;0]; % BCs for T at x3=L
[C1sol,C2sol,C3sol,C4sol] = solve([BC1,BC2],[C1,C2,C3,C4]); % Solve for the unknown consts
u1 = simplify(subs(u1,[C1,C2,C3,C4,x3],[C1sol,C2sol,C3sol,C4sol,L])) % u1 at x3=L
u2 = simplify(subs(u2,[C1,C2,C3,C4,x3],[C1sol,C2sol,C3sol,C4sol,L])) % u2 at x3=L
```

```matlab
% Script to calculate deflections of a beam
% You can easily adapt this script to other loads and boundary conditions
syms EE I11 I22 I12 M T1 T2 T3 a t L P x3 C1 C2 C3 C4 w real
syms u1(x3) u2(x3)
% The second moments of area
I11 = II(1,1);
I12 = -II(1,2); % A positive off diagonal means I12 is a negative number
I22 = II(2,2);
% The two differential equations
diffeq1 = EE*(I22*diff(u1(x3),x3,4)+I12*diff(u2(x3),x3,4))==0;
diffeq2 = EE*(I12*diff(u1(x3),x3,4)+I11*diff(u2(x3),x3,4))==-w;
% Zero displacement and slope at x3=0
BC1 = u1(0)==0;
BC2 = u2(0)==0;
BC3 = subs(diff(u1,x3),x3,0)==0;
BC4 = subs(diff(u2,x3),x3,0)==0;
BC5 = u1(L) == C1;
BC6 = u2(L) == C2;
BC7 = subs(diff(u1(x3),x3),x3,L)==C3;
BC8 = subs(diff(u2(x3),x3),x3,L)==C4;
% The solution with displacements/slopes prescribed at both ends
[u1,u2] = dsolve([diffeq1,diffeq2],[BC1,BC2,BC3,BC4,BC5,BC6,BC7,BC8]);
% Now calculate the bending moments and transverse forces
M1 = simplify(EE*(I22*diff(u1,x3,2)+I12*diff(u2,x3,2)));
M2 = EE*(I12*diff(u1,x3,2)+I11*diff(u2,x3,2));
T1 = -EE*(I22*diff(u1,x3,3)+I12*diff(u2,x3,3));
T2 = -EE*(I12*diff(u1,x3,3)+I11*diff(u2,x3,3));
% And set up the boundary conditions we really want at x3=L
BC5 = subs(M1,x3,L)==0;
BC6 = subs(M2,x3,L)==0;
BC7 = subs(T1,x3,L)==0;
BC8 = subs(T2,x3,L)==0;
% Use the correct BCs to eliminate C1-C4
[C1sol,C2sol,C3sol,C4sol] = solve([BC5,BC6,BC7,BC8],[C1,C2,C3,C4]);
% Substitute back into the solution
u1 = simplify(subs(u1,[C1,C2,C3,C4,x3],[C1sol,C2sol,C3sol,C4sol,L]))
u2 = simplify(subs(u2,[C1,C2,C3,C4,x3],[C1sol,C2sol,C3sol,C4sol,L]))
```

u1 =

$$-\frac{L^4 w \, (\pi - 4)}{EE \, a^3 t \, (\pi^3 - 20\pi + 32)}$$

u2 =

$$-\frac{L^4 w \, (\pi^2 - 8)}{2 \, EE \, a^3 t \, (\pi^3 - 20\pi + 32)}$$

Another (and for this particular problem, simpler) approach is to calculate the bending moment distribution by hand

$$M_2 = \frac{wx_3^2}{2} \qquad M_1 = 0$$

$$\Rightarrow \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \frac{d^2}{dx_3^2} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} wx_3^2/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{d^2}{dx_3^2} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \frac{1}{E(I_{11}I_{22} - I_{12}^2)} \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \begin{bmatrix} wx_3^2/2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{d^2u_2}{dx_3^2} = -\frac{I_{22}}{E(I_{11}I_{22} - I_{12}^2)} \frac{wx_3^2}{2} \qquad \frac{d^2u_1}{dx_3^2} = \frac{I_{12}}{E(I_{11}I_{22} - I_{12}^2)} \frac{wx_3^2}{2}$$

$$\Rightarrow u_2 = -\frac{I_{22}}{E(I_{11}I_{22} - I_{12}^2)} \frac{wx_3^4}{24} \qquad u_1 = \frac{I_{12}}{E(I_{11}I_{22} - I_{12}^2)} \frac{wx_3^4}{24}$$

[5 POINTS]
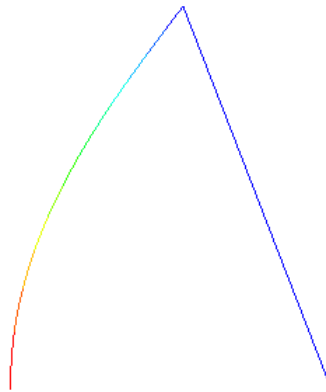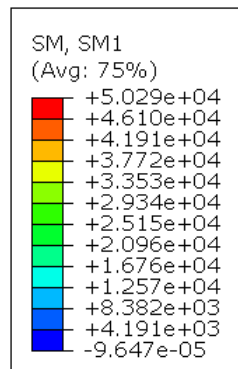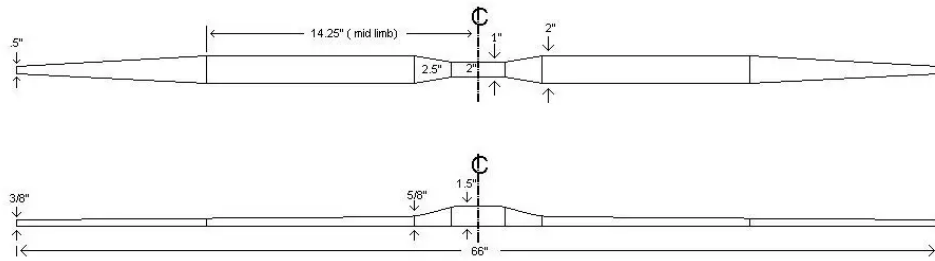
5. In this problem you will explore the beam elements available in ABAQUS by setting up a simple finite element simulation to predict the static force-v-draw curve for an English longbow. Dimensions of a simple home-made bow (from this reference) are shown in the figure.

5.1 Because of symmetry, we only need to model half the bow. Create a 3D deformable part with wire base feature, with one end at the origin and the other at y= 830mm (the undeformed bow is straight; it gets bent when the string is attached). Note that this procedure will compute *half* the total draw force (there will be an additional force from the bottom half of the bow that has not been modeled).

5.2 Use a Young's modulus for Yew wood of 6.5 GPa (convert to N/mm^2 since lengths are in mm), and make the cross-section a 40mmx12mm rectangle (it is possible to create tapered cross-sections in ABAQUS/CAE but you have to define the taper of every element in the beam separately, which is extremely tedious to do by hand and best done with a python script). Don't forget to assign a beam section orientation – make sure the normal to the flat side of the bow is parallel to the x direction.

5.3 Create a second part for the bow string. Make one end of the string coincident with the tip of the bow, and make the string 40mm shorter than the bow. Give the bow string a modulus of 110 GPa and a circular cross section with radius 0.72mm.

5.4 Create two steps, both with 1 sec duration – these will be used to string the bow, and then draw it. Make sure NLGEOM is selected for both steps. Make the initial and maximum increment size 0.1s for both steps. In the step module for the first step, open the Field Output Request manager and add requests to plot the

section forces/moments.   Also, (i) create a new Set (use Tools > Set > Create… name the set drawpoint, Select Geometry, and select the bottom end of the bowstring); then (ii) create a history output request to request displacements/velocity/acceleration and forces/reactions for this set (this will be used to plot the force-v-draw curve).

5.5 To tie the bowstring to the bow, select Constraint > Create …, select a tie constraint; in the menu below the viewport select Node Region, then select the end points of the string and bow as the master and slave (it doesn't matter which is which).   If you need to, you can right click a part under the 'instance' branch of the model tree to hide it.   In the 'edit constraint' menu uncheck the 'Tie rotational DOFs if applicable' option

5.6 In the boundary condition module, create a BC for the 'initial' step that makes all the degrees of freedom (U and UR) to zero for the node on the bow that lies at the origin. This BC will be applied throughout the analysis by default.

5.7 We need to define boundary conditions for the two load steps.   The first step will be used to string the bow.   To do this we need to apply a force to bend the bow, and pull the string down so its bottom end lies at y=0. Apply a horizontal force (acting in the positive x direction) to the bow at its tip with magnitude 20 N (if your bow is vertical, the force will be horizontal).   Click the $\sim$ button next to the drop down menu to define an amplitude for this force; check the 'tabular' radio button, and enter a table that starts at 1 at time t=0 and ends at 0 at time t=1.   Then select Amp-1 in the dropdown menu. This will bend the bow so it can be pulled down by the bowstring, but will remove the force by the end of the step. Next, create a BC that will apply a displacement to the free end of the bowstring that will pull it down to be level with the base of the bow (this will be U2=-40mm). A standard 'Ramp' amplitude will work. Finally open both the Load and BC managers and deactivate the two BCs created for step 1 in step 2.

5.8 To draw the bow, (i) create a BC that will fix U2 at the bottom of the bowstring to remain constant in the second step. To do this open the BC manager, and make sure the BC you defined at the end of the string in Step 1 is deactivated in Step 2.   Then create a new BC, select the bottom end of the bowstring, select 'Fixed at Current Position' in the Edit BC window, and check the box for U2.   Next , create a second BC that will apply a constant horizontal velocity (V1) of 200 mm/sec to the bottom end of the bowstring.   This will draw the bow by 200mm by the end of the step.

5.9 In the mesh module seed the bow and string with 10mm/20mm mesh size, respectively, and mesh both parts with the default element type.

5.10 Submit and run the job.   If all works correctly you should see a deformed shape and bending moment distribution that looks something like the picture on the previous page.

5.11 Create and hand in a plot of the force-v-draw curve for the bow.   Work out how to correct the draw force to double the reaction force.   For a rough comparison, you can find measured force-v-draw curves for a few bows on this page – but those bows are made from different materials and have a different cross-section to the one analyzed here (which seems to be a rather light bow).

[10 POINTS]