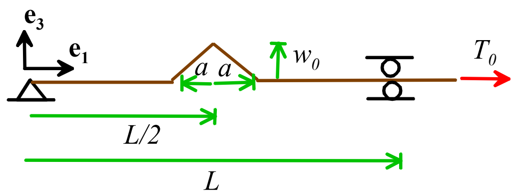
School of Engineering
Brown University

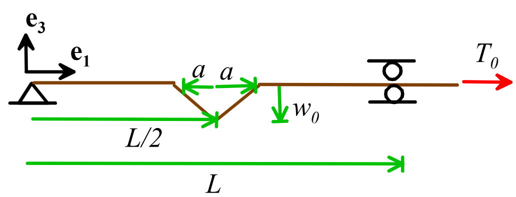
EN1750: Advanced Mechanics of Solids

Homework 9 Dynamics.

Due Friday Nov 29, 2019

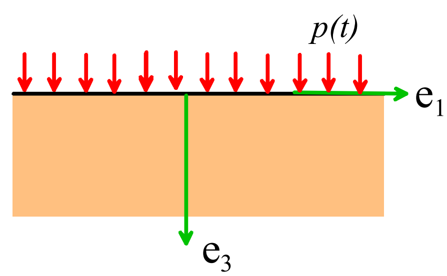
1. The string shown in the figure has mass per unit length m and is stretched by an axial tension T_0. It is prevented from moving in a vertical direction at its ends $x_3 = 0, x_3 = L$. At time *t=0* it is released from rest with the displacement distribution shown in the figure. Draw the shape of the string at time $t = L\sqrt{m/T_0}$





[3 POINTS]

2. A linear elastic half-space with Young's modulus E and Poisson's ratio ν is stress free and stationary at time *t=0*, is then subjected to a constant pressure p_0 on its surface for *t>0*.



2.1 Calculate the stress, displacement and velocity in the solid as a function of time (you can express your answer in terms of p_0, t, x_3, ρ and the wave speed c_L)?

From class we know that

$$\sigma_{33} = \begin{cases} -p_0 & t - x_3/c_L > 0 \\ 0 & t - x_3/c_L > 0 \end{cases}$$

$$v_3 = \begin{cases} \dfrac{p_0}{\rho c_L} & t - x_3/c_L > 0 \\ 0 & t - x_3/c_L > 0 \end{cases}$$

We also know $\varepsilon_{11} = \varepsilon_{22} = 0$ so the elastic stress-strain relations give

$$\sigma_{11} = \sigma_{22} = \frac{\nu}{(1-\nu)}\sigma_{33}$$

We can integrate the velocity to find the displacement

$$u_3 = \begin{cases} \dfrac{p_0}{\rho c_L}(t - x_3 / c_L) & t - x_3 / c_L > 0 \\ 0 & t - x_3 / c_L > 0 \end{cases}$$

[3 POINTS]

2.2 Calculate the total kinetic energy (per unit area of the surface) of the half-space as a function of time

The kinetic energy is $\int_0^{tc_L} \frac{1}{2}\rho v_3^2 dx_3 = \frac{1}{2}\rho\left(\frac{p_0}{\rho c_L}\right)^2 tc_L = \frac{1}{2}\frac{p_0^2 t}{\rho c_L}$

[2 POINTS]

2.3 Calculate the total strain energy of the half-space (per unit area of surface) as a function of time

The strain energy density is

$$\int_0^{tc_L} \frac{1}{2}\sigma_{33}\varepsilon_{33}dx_3 = \int_0^{tc_L} \frac{1}{2}(-p_0)\left(\frac{-p_0}{\rho c_L^2}\right) = \frac{1}{2}\frac{p_0^2 t}{\rho c_L}$$

[2 POINTS]

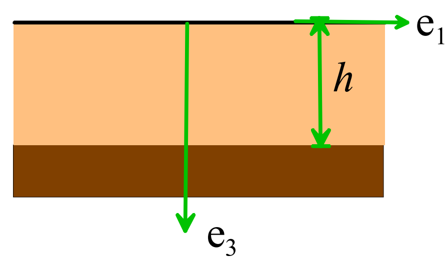
2.4 Verify that the sum of the strain energy and kinetic energy is equal to the work done by the tractions acting on the surface of the half-space.

The work done (per unit area) by the surface traction is $\int_0^t v_3 p_0 = \frac{p_0^2 t}{\rho c_L}$

This is equal to the sum of strain energy and kinetic energy, as expected.

[2 POINTS]

3. This paper describes a design for a solid mounted resonator. It consists of a thin piezoelectric film that is deposited on the surface of a 'Bragg reflector' – which can be idealized as a rigid boundary. The goal of this problem is to calculate a formula for the frequency of the 5[th] vibration mode (through-thickness vibrations). Assume that the resonator is a thin film with thickness *h*, Youngs modulus E, Poisson's ratio ν, and mass density ρ; and that the displacement in the plate has the form $u_3 = u(x_3)$, with all other components zero.



3.1 Show that the equation of motion for *u* reduces to

$$\frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2}$$

and give a formula for c_L

Follow the derivation of the plane wave solutions in class

$$\sigma_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial u}{\partial x_3}$$

The linear momentum balance equation then gives

$$\frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x_3^2} = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial t^2}$$

With $c_L = \sqrt{E(1-\nu)/\rho(1+\nu)(1-2\nu)}$

[3 POINTS]

3.2 Assume that the top surface of the plate is stress free, and the bottom is fixed. Write down the boundary condition for *u* at $x_3 = 0, x_3 = h$.

The zero stress boundary condition means that the through-thickness strain must be zero at the surface:

$$\frac{\partial u}{\partial x_3} = 0 \qquad x_3 = h$$

At the base of the film *u=0*.

[1 POINT]

3.3 Consider solutions to the equation of motion of the form $u = \cos(\omega t + \phi) f(x_3)$. Use 3.1 to find an ODE for $f(x_3)$. Find the general solution for *f* along with the formula relating wave number *k* to frequency ω (the dispersion relation…)

Substitute into the governing equation

$$\frac{\partial^2 f}{\partial x_3^2}\cos(\omega t+\phi) = -\frac{\omega^2}{c_L^2} f\cos(\omega t+\phi) \Rightarrow \frac{\partial^2 f}{\partial x_3^2} + k^2 f = 0$$

This has general solution $f = A\sin kx_3 + B\cos kx_3$ with $k = \omega / c_L$

[2 POINTS]

3.4 Show that the boundary conditions can be expressed in matrix form as

$$\begin{bmatrix} \cos kh & -\sin kh \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, find a formula for the resonant wave numbers k and the corresponding resonant frequencies ω

The boundary condition reduces to $\frac{df}{dx_3} = Ak\cos kx_3 - Bk\sin kx_3 \quad x_3 = h$

Together with $f = A\cos kx_3 + B\sin kx_3 = 0$

For nontrivial solutions the determinant of the matrix must vanish, which gives $\cos(kh) = 0$. This has solutions $kh = \frac{\pi}{2} + n\pi \Rightarrow \omega = \frac{(2n+1)\pi c_L}{2h}$

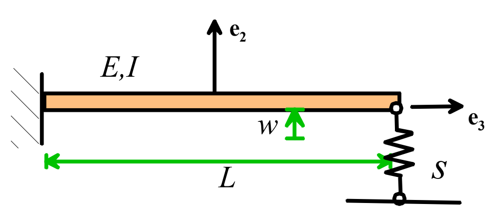
[2 POINTS]

3.5 Calculate the thickness of a resonator made from PZT with Young's modulus 81GPa and Poissons ratio 0.39 and mass density 7320 kg/m^3 with a resonant frequency of 10 MHz.

The numbers give a longitudinal wave speed of 4730m/s. The necessary thickness is $\pi c_L / 2\omega$ =0.12mm. (don't forget the factor of 2π to calculate the angular frequency…)

[1 POINT]

4. This publication analyzes the vibration of the cantilever in an atomic force microscope. The goal of this problem is to repeat some of their calculations. The figure shows the problem to be solved: the cantilever is idealized as a beam with modulus E and mass moment of inertia I, mass density ρ and cross sectional area A. The interaction of the microscope tip with the surface of the specimen is approximated by a spring with stiffness s (we use s instead of the usual k because k is used for the wave number). Assume that the spring is free of force when the displacement of the cantilever is zero.



4.1 Write down the differential equation governing flexural vibration of the cantilever, and by considering solutions of the form $w = \cos(\omega t+\phi) f(x_3)$ show that the equation is satisfied by a solution of the form

$$f(x_3) = A\sin kx_3 + B\cos kx_3 + C\sinh kx_3 + D\cosh kx_3$$

(you can use exponential solutions of the form $f(x_3) = \sum_{i=1}^{4} \exp(\lambda_i x_3)$ if you prefer, where λ_i are the roots of the characteristic equation, but messing with the complex numbers is a bit painful). Write down the relationship between $k, \omega,$ and $\beta = \sqrt{EI / \rho A}$

The differential equation is

$$\frac{d^4 w}{dx_3^4} + \frac{1}{\beta^2}\frac{d^2 w}{dt^2} = 0$$

Substituting the solution gives

$$\frac{d^4 f}{dx_3^4} - \frac{\omega^2}{\beta^2} f = 0$$

The given solution for *f* satisfies the equation with

$$k^4 = \frac{\omega^2}{\beta^2}$$

[2 POINTS]

4.2 Write down the boundary conditions for the transverse displacement *w*, and show that they can be arranged into the following form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kL) & -\cos(kL) & \sinh(kL) & \cosh(kL) \\ -(kL)^3\cos(kL) - \mu\sin(kL) & (kL)^3\sin(kL) - \mu\cos(kL) & (kL)^3\cosh(kL) - \mu\sinh(kL) & (kL)^3\sinh(kL) - \mu\cosh(kL) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mu = \dfrac{sL^3}{EI}$

The boundary conditions are

1. Zero deflection at $x_3 = 0$, which requires $B + D = 0$
2. Zero slope at $x_3 = 0$, which requires $A + C = 0$
3. Zero bending moment at $x_3 = L$, i.e. $d^2 w / dx_3^2 = 0$
   This gives $(-A\sin kL - B\cos kL + C\sinh kL + D\cosh kL)$
4. The transverse force must satisfy $T_2 = -kw$ at $x_3 = L$ which requires

$$-EI\frac{d^3 w}{dx_3^3} = -sw$$

$$\Rightarrow -EIk^3\left(-A\cos kL + B\sin kL + C\cosh kL + D\sinh kL\right) = -s(A\sin kL + B\cos kL + C\sinh kL + D\cosh kL)$$

We can re-write this as

$$(kL)^3\left(-A\cos kL + B\sin kL + C\cosh kL + D\sinh kL\right) = \frac{sL^3}{EI}(A\sin kL + B\cos kL + C\sinh kL + D\cosh kL)$$

This can be re-written as

$$A(-(kL)^3\cos kL - \mu\sin kL) + B((kL)^3\sin kL - \mu\cos kL) +$$
$$C((kL)^3\cosh kL - \mu\sinh kL) + D((kL)^3\sinh kL - \mu\cosh kL) = 0$$

Collecting all four boundary conditions into matrix form gives

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kL) & -\cos(kL) & \sinh(kL) & \cosh(kL) \\ -(kL)^3\cos(kL) - \mu\sin(kL) & (kL)^3\sin(kL) - \mu\cos(kL) & (kL)^3\cosh(kL) - \mu\sinh(kL) & (kL)^3\sinh(kL) - \mu\cosh(kL) \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Alternatively we can have MATLAB do the heavy lifting

```
clear all
syms k L mu x3 omega t
w = [sin(k*x3),cos(k*x3),sinh(k*x3),cosh(k*x3)];
N = [subs(w,x3,0);...
     subs(diff(w,x3),x3,0);...
     subs(diff(w,x3,2),x3,L);...
     simplify(subs( (-mu*w+(L)^3*diff(w,x3,3)),x3,L) )]
```

[5 POINTS]

4.3 Hence, show that the wave numbers for the vibration modes satisfy

$$(kL)^3\cos(Lk)\cosh(Lk) + \mu\cosh(Lk)\sin(Lk) - \mu\cos(Lk)\sinh(Lk) + (kL)^3 = 0$$

Use Matlab…

```
clear all
syms k L mu x3 omega t
w = [sin(k*x3),cos(k*x3),sinh(k*x3),cosh(k*x3)];
N = [subs(w,x3,0);...
     subs(diff(w,x3),x3,0);...
     subs(diff(w,x3,2),x3,L);...
     simplify(subs( (-mu*w+(L)^3*diff(w,x3,3)),x3,L) )]
 simplify(det(N))
```

This gives the answer stated

[2 POINTS]

4.4 Calculate the lowest natural frequency of the beam (in terms of β and L) without the spring on its end (i.e. $\mu = 0$). You will need to solve the equation $\cos(Lk)\cosh(Lk) + 1 = 0$ numerically, eg using fsolve in MATLAB. You can google the answer to check it; this is a well known result.

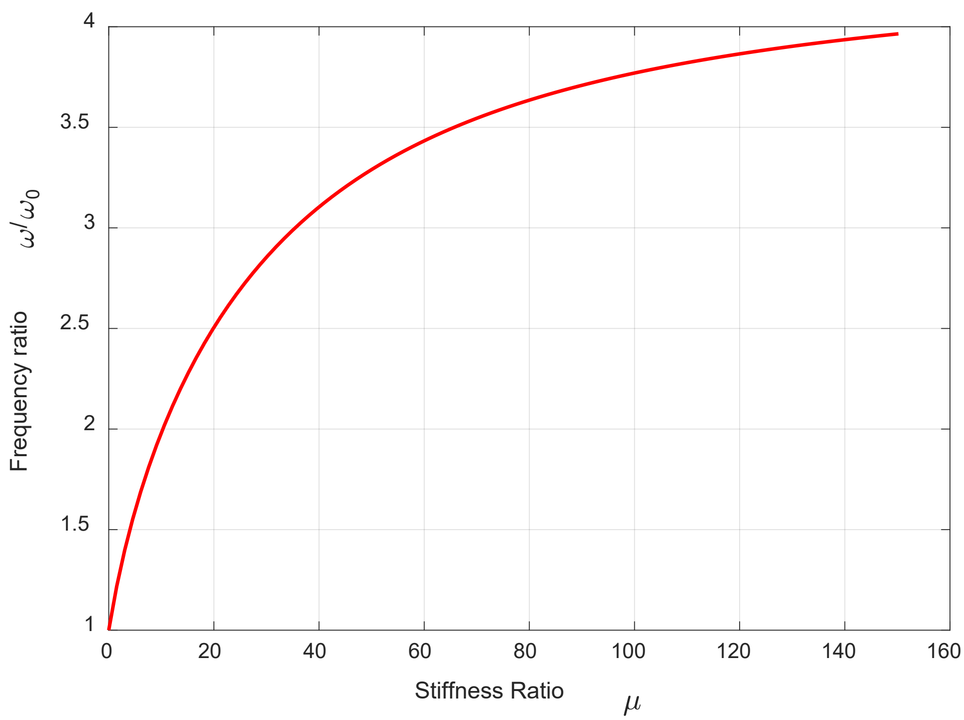
Solving the equation gives $kL = 1.875$, the dispersion relation gives

$$\omega = k^2 \beta = \frac{(1.875)^2}{L^2} \beta$$

[2 POINTS]

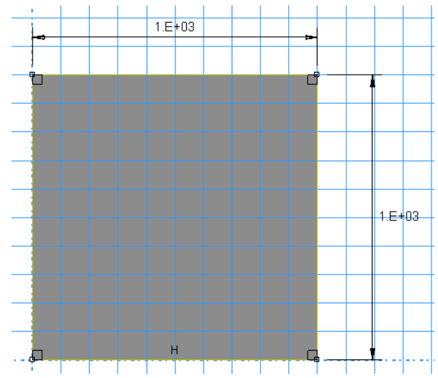
4.5. Plot a graph of $\omega(\mu)/\omega(0)$, where $\omega(0)$ is the frequency of the cantilever without the spring on its end (i.e. the solution to 4.4) as a function of $0 < \mu < 150$. You'll need to write a short MATLAB script to do this.

The graph is plotted below.



[3 POINTS]

5. In this problem you will use explicit dynamic simulations in ABAQUS to study wave propagation near the surface of an elastic solid.



5.1 Create a 2D planar part like the one shown in the figure. The dimension unit is millimeters.

5.2 Create a material with Youngs modulus 100 GPa, Poissons ratio 0.3 and mass density 10000 kg/m^3. We will use N for forces and mm for length (so stresses are in N/mm^2 = MN/m^2) – this means 100GPa should be entered as 100000MPa. We need to use a mass density that is consistent with these units. Convince yourself that if we choose to use N for force, mm for length, and milliseconds for time, we must enter density in N milliseconds^2/mm^4, which makes $\rho = 10\times10^{-3}$ in our chosen unit system (note that wave speeds in m/s and mm/millisecond are identical). Assign the part a homogeneous section with these properties.

5.3 Create an instance of the part in the usual way.

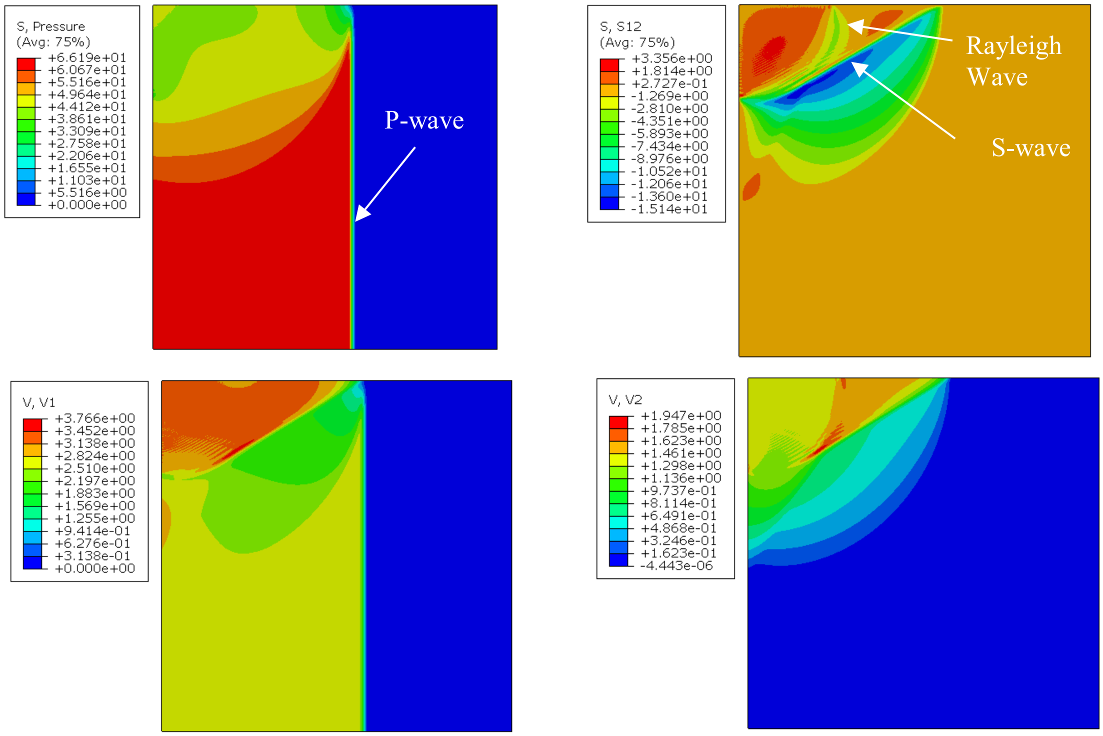
5.4 Calculate the expected speeds of P, S, and Rayleigh waves in the material. Hence, estimate the time (in milliseconds) required for each wave to propagate from one side of the solid to the other. Then create an Explicit Dynamic step with duration roughly twice the time required for the pressure wave to propagate across the solid (enter the time in milliseconds, since that's our chosen time unit.). You can turn off NLGEOM for this problem. Use Output->Field Output requests to edit the field output. Change the frequency of output to make sure that at least 200 frames are saved during the analysis (you can change this – you'll get nicer animations with more frequent output, but it will produce huge odb files and slow down the analysis).

5.5 Enter a BC to prevent vertical motion of the base of the block, and apply an instantaneous pressure of 100MPa on the left face.

5.6 Assign an element type of Plane Strain reduced integration elements (CPE4R) from the Explicit element library. Note that the default is plane stress elements so it's important to select the right options in the Element Type menu. Seed the part with a 1.5mm mesh size (if you don't mind waiting a bit longer for the analysis to complete you could try 1mm) and mesh it with quad elements (use a structured meshing algorithm).

5.7 Submit and run the job. Use Job-> Monitor to track the progress of the analysis – note that since this is an explicit dynamic simulation ABAQUS takes a very large number of very small time-steps.

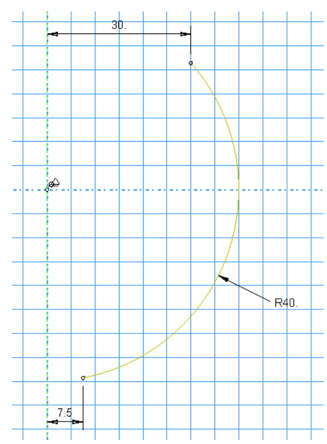
5.8 You can use the visualization module to watch some fun movies of wave propagation and reflection in the block; you can hopefully figure out what you can see! As a submission for this problem, please plot contours of (i) pressure; (ii) shear stress S12; (iii) velocity V1; and (iv) velocity V2 at a time of around 0.15 milliseconds. Mark on your plots (a) a plane P-wave; (b) a plane S-wave and (c) a Rayleigh wave (these will show up as changes in stress – for example we expect the pressure to jump across a P-wave but not an S wave; and we will see changes in velocity across any plane wave front). You can double check the wave types by calculating the velocity of each type of wave.



[10 POINTS]

6. In this problem you will calculate the resonant frequencies of an (approximate) wine glass.

6.1 Create a 3D deformable part with shell/revolution base feature. Use something like the sketch shown in the figure (dimensions are in mm), and revolve it through 360 degrees about the centerline to create the glass (fixed BCs will be applied to the small hole at the bottom to represent the stem).

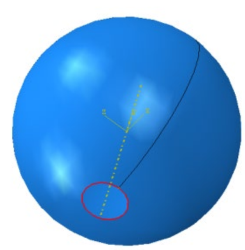


6.2 Create a material with Youngs modulus 10GPa, Poissons ratio 0.2 and mass density 2700 kg/m^3 Note that you will have to choose a unit system with lengths in mm – you could use the procedure suggested in the previous problem to do this, but your frequencies will then be reported in cycles/millisecond instead of Hz. You could use forces in N, lengths in mm and time in sec if you prefer but you will need to figure out how to change the density to make this work! Create a homogeneous shell section with 0.8mm thickness and assign it to the part.

6.3 Create an instance of the part

6.4 Create a Linear Perturbation/Frequency step and request 50 or so eigenmodes (you can do fewer if you are in a hurry)

6.5 For BCs, fix all DOF on the bottom edge of the bowl (see the figure)



6.6 Assign a quadratic quad element type with 6DOF per node to the part, seed it with a mesh size 2mm, and mesh it with Sweep algorithm.

6.7 Create/Run the job, and check the mode shapes/frequencies in the visualization module. For comparison, this video measures a frequency for the 3rd mode (the lowest two are hard to excite with sound) of 317Hz… Hand in a plot showing the mode shapes and frequencies for the first, 3rd, and 15th modes.

[10 POINTS]