EN1750: Advanced Mechanics of Solids

Example problems on plasticity and failure.

Ungraded

School of Engineering
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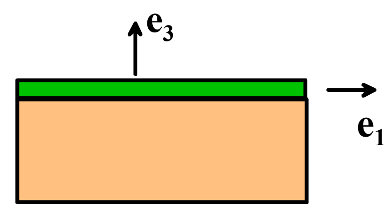


1. An isotropic, elastic-perfectly plastic thin film with Young's Modulus E, Poisson's ratio ν, yield stress in uniaxial tension Y and thermal expansion coefficient α is bonded to a stiff substrate. It is stress free at some initial temperature and then heated. The substrate prevents the film from stretching in its own plane, so that $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{12} = 0$, while the surface is traction free, so that the film deforms in a state of plane stress. Calculate the critical temperature change ΔT_y that will cause the film to yield, using (a) the Von Mises yield criterion and (b) the Tresca yield criterion.

The film is in a state of plane stress. We can use the plane stress relations to calculate the stress in the film (assuming elastic behavior)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} - \frac{E\alpha\Delta T}{(1-\nu)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The strains are zero, so the total stress is $\sigma_{11} = \sigma_{22} = -E\alpha\Delta T/(1-\nu)$ with all other stress components zero.

The von-Mises stress is $\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right\}} = E\alpha\left|\Delta T\right|/(1-\nu)$

The yield criterion is $\sigma_e = Y \Rightarrow \Delta T_y = \pm Y(1-\nu)/\alpha E$

The Tresca yield criterion is $\max\left\{\left|\sigma_1-\sigma_2\right|, \left|\sigma_1-\sigma_3\right|, \left|\sigma_2-\sigma_3\right|\right\} = Y$ which gives the same answer

2. Assume that the thin film described in the preceding problem shows so little strain hardening behavior that it can be idealized as an elastic-perfectly plastic solid, with uniaxial tensile yield stress Y. Suppose the film is stress free at some initial temperature, and then heated to a temperature $\beta\Delta T_y$, where ΔT_y is the yield temperature calculated in the preceding problem, and $\beta > 1$.

2.1 Find the stress in the film at this temperature

The film must remain in a state of plane stress, and will deform plastically as it is heated further.

We know that the two in-plane stresses must be equal by symmetry, so we can set $\sigma_{11}=\sigma_{22}=\sigma$. The Von-Mises (or Tresca) yield criterion shows that

$$\sigma_e=\sqrt{\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right\}}=|\sigma|=Y$$

Therefore $\sigma=\pm Y$.
The stresses can't jump from tension to compression as the film is heated through its yield point, so the stresses must be compressive. So $\sigma_{11}=\sigma_{22}=-Y$.

2.2 The film is then cooled back to its original temperature. Find the stress in the film after cooling.

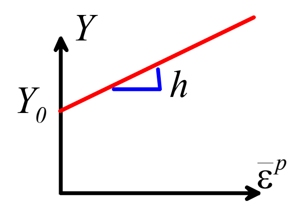
During cooling, the stresses will be reduced, so the film (at least at first) will unload elastically. We can calculate the change in stress during unloading using the answer to problem 1.

$$\sigma_{11}=\sigma_{22}=-Y+E\alpha\Delta T/(1-\nu)=-(1-\beta)Y$$

This answer is valid as long as the new stresses don't exceed yield. We can repeat the calculation in problem 1 to see that the stresses will exceed yield after cooling if $\beta>2$. So

$$\sigma_{11}=\sigma_{22}=\begin{cases}-(1-\beta)Y & \beta<2\\ Y & \beta>2\end{cases}$$

3. Suppose that the thin film described in the preceding problem is made from an elastic, isotropically hardening plastic material with a Mises yield surface, and yield stress-v-plastic strain as shown in the figure. The film is initially stress free, and then heated to a temperature $\beta\Delta T_y$, where ΔT_y is the yield temperature calculated in problem 1, and $\beta>1$.



3.1 Find a formula for the stress in the film at this temperature. To do this you will need to assume the film remains in a state of plane biaxial stress $\sigma_{11}=\sigma_{22}=\sigma$ with all other stresses zero, and then

(i) Find formulas for the hydrostatic stress, deviatoric stress and Von Mises stress in terms of the unknown stress

(ii) Write down an expression for the total strain rate (elastic, plastic and thermal) in terms of the stress rate, using the expressions from class

(iii) Use the condition that the total strain rate is zero to relate the stress rate to the rate of change of temperature

(iv) Integrate the result of (iii) with time (with initial condition $\sigma=-Y$ when $\Delta T=\Delta T_y$) to find the stress.

Following the stated steps, we assume that the only nonzero stresses are $\sigma_{11}=\sigma_{22}$ and have some unknown magnitude σ

The hydrostatic stress is therefore $\frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33})=\frac{2}{3}\sigma$

The deviatoric stress is $S_{ij}=\sigma_{ij}-\sigma_{kk}\delta_{ij}/3\Rightarrow S_{11}=S_{22}=\frac{1}{3}\sigma \qquad S_{33}=-\frac{2}{3}\sigma$

The Von-Mises stress is (from the previous problem) $\sigma_e = |\sigma|$

The formula for the total strain rate is

$$\frac{d\varepsilon_{ij}}{dt} = \frac{d\varepsilon_{ij}^e}{dt} + \frac{d\varepsilon_{ij}^p}{dt} + \frac{d\varepsilon_{ij}^T}{dt} = \frac{1+\nu}{E}\left(\frac{d\sigma_{ij}}{dt} - \frac{\nu}{1+\nu}\frac{d\sigma_{kk}}{dt}\delta_{ij}\right) + \frac{3}{2}\frac{\left\langle S_{kl}\frac{d\sigma_{kl}}{dt}\right\rangle}{h\sigma_e}\frac{3}{2}\frac{S_{ij}}{\sigma_e} + \frac{d\Delta T}{dt}\alpha\delta_{ij}$$

We know $\varepsilon_{11} = \varepsilon_{22}$ so we only need to calculate two components of strain rate

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{33}\end{bmatrix} = \frac{1}{E}\frac{d}{dt}\begin{bmatrix}(1-\nu)\sigma\\ -2\nu\sigma\end{bmatrix} + \frac{3}{2}\frac{(\sigma/3+\sigma/3)(d\sigma/dt)}{h\sigma^2}\frac{3}{2}\begin{bmatrix}\sigma/3\\ -2\sigma/3\end{bmatrix} + \alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\ 1\end{bmatrix}$$

Since $\varepsilon_{11} = 0$ we see that

$$\left(\frac{1-\nu}{E} + \frac{1}{2h}\right)\frac{d\sigma}{dt} = -\alpha\frac{d\Delta T}{dt}$$

We can integrate this (with the condition that $\sigma = -Y_0$ when $\Delta T = Y_0(1-\nu)/\alpha E = \Delta T_y$) to see that

$$\sigma = -Y_0 - \frac{2\alpha h E(\beta-1)\Delta T_y}{E+2(1-\nu)h} = -Y_0 - Y_0\frac{2h(1-\nu)(\beta-1)}{E+2(1-\nu)h} = -Y_0\frac{E+2h(1-\nu)\beta}{E+2(1-\nu)h}$$

3.2 The film is then cooled back to its original temperature. Find the stress in the film after cooling.

The unloading is elastic. We can find the change in stress from the solution to problem 2.2, so that

$$\sigma = -Y_0\frac{E+2h(1-\nu)\beta}{E+2(1-\nu)h} + \frac{E\alpha\Delta T}{(1-\nu)} = -Y_0\frac{E+2h(1-\nu)\beta}{E+2(1-\nu)h} + \beta Y_0 = \frac{E(\beta-1)Y_0}{E+2h(1-\nu)}$$

3.3 The film is cooled further by a temperature change $\Delta T < 0$. Calculate the critical value of ΔT that will cause the film to reach yield again.

The yield stress during heating increased to the value given in 3.1, i.e.

$$Y = Y_0\frac{E+2h(1-\nu)\beta}{E+2(1-\nu)h}$$

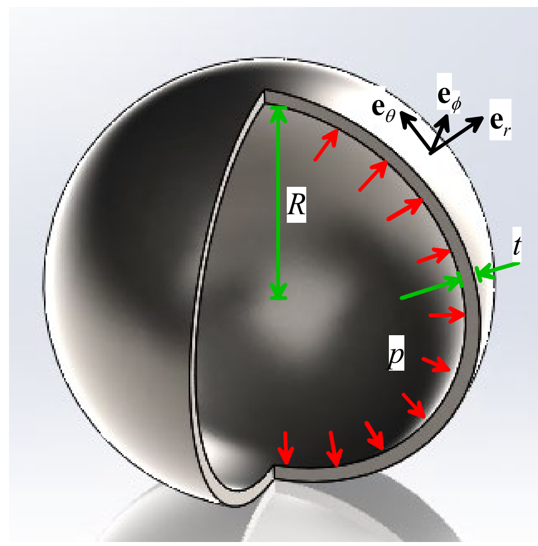
The stress after further cooling will increase to

$$\sigma = \frac{E(\beta-1)Y_0}{E+2h(1-\nu)} + \frac{E\alpha\Delta T}{(1-\nu)}$$

The film will yield when $\sigma = Y$, which gives

$$Y_0 \frac{E+2h(1-\nu)\beta}{E+2(1-\nu)h} - \frac{E(\beta-1)Y_0}{E+2h(1-\nu)} = \frac{E\alpha\Delta T}{(1-\nu)}$$
$$\Rightarrow Y_0 \frac{E(2-\beta)+2h(1-\nu)\beta}{E+2(1-\nu)h} = \frac{E\alpha\Delta T}{(1-\nu)}$$
$$\Rightarrow \Delta T = \frac{(1-\nu)Y_0}{E\alpha}\left(\frac{E(2-\beta)+2h(1-\nu)\beta}{E+2(1-\nu)h}\right)$$

4. A thin-walled sphere with radius R and wall thickness t is made from an elastic-plastic material with Youngs modulus E , Poissons ratio ν and a linear hardening relation $Y = Y_0 + h\varepsilon_e$. The sphere is subjected to monotonically increasing internal pressure p (with $dp/dt > 0$), which generates a stress state (in spherical-polar coordinates) $\sigma_{rr} \approx 0,\ \sigma_{\phi\phi} = \sigma_{\theta\theta} = pR/(2t)$ (note that these are principal stresses)



4.1 Find a formula for the Von-Mises stress in the sphere wall, in terms of p,R and t. Hence, calculate the pressure that will first cause yield in the sphere wall.

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]} = pR/2t$$

At yield $\sigma_e = Y \Rightarrow p_y = 2tY/R$

4.2 Find the hydrostatic and deviatoric stresses in the sphere wall.

$$\sigma_h = \sigma_{kk}/3 = pR/3t$$

$$S_{rr} = -pR/3t \qquad S_{\theta\theta} = S_{\phi\phi} = pR/6t$$

4.3 Hence, find a formula for the Von Mises plastic strain rate $d\varepsilon_e/dt$ in the sphere wall, in terms of $dp/dt, h, R, t$

From notes, the plastic strain rate is zero below yield while above yield, the formula is

$$\frac{d\varepsilon_e}{dt} = \frac{3}{2}\frac{1}{h\sigma_e}\left\langle S_{ij}\frac{d\sigma_{ij}}{dt}\right\rangle = \frac{3}{2}\frac{1}{h(pR/2t)}\left\langle 2\frac{pR}{6t}\frac{dp}{dt}\frac{R}{2t}\right\rangle = \frac{1}{h}\frac{R}{2t}\frac{dp}{dt}$$

4.4 Hence, find a formula for the total strain rates $d\varepsilon_{rr} / dt, d\varepsilon_{\theta\theta} / dt$ (include both elastic and plastic strain rates, and give solutions for pressure both below and above yield) in the shell.

From notes

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\end{bmatrix} = \begin{cases} \dfrac{R}{2Et}\begin{bmatrix}-2\nu\\ 1-\nu\end{bmatrix}\dfrac{dp}{dt} & pR/(2t) < Y_0 \\ \dfrac{R}{2Et}\begin{bmatrix}-2\nu\\ 1-\nu\end{bmatrix}\dfrac{dp}{dt} + \dfrac{1}{h}\dfrac{R}{2t}\dfrac{dp}{dt}\begin{bmatrix}-1\\ 1/2\end{bmatrix} & pR/(2t) < Y_0 \end{cases}$$

4.5 Find the total hoop strains $\varepsilon_{\theta\theta}, \varepsilon_{\phi\phi}$ when the pressure reaches a value $p = 4tY_0 / R$

Integrating

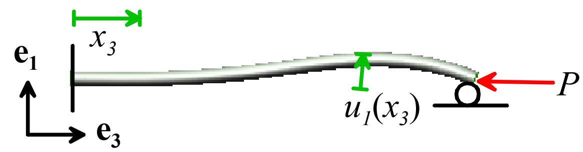
$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\end{bmatrix} = \frac{Y_0}{E}\begin{bmatrix}-2\nu\\ 1-\nu\end{bmatrix} + \frac{R}{2t}\begin{bmatrix}-2\nu/E - 1/h\\ (1-\nu)/E + 1/(2h)\end{bmatrix}\left(p - 2Y_0 t / R\right)$$
$$= Y_0\begin{bmatrix}-4\nu/E - 1/h\\ 2(1-\nu)/E + 1/(2h)\end{bmatrix}$$

4.6 Find a formula for the change in radius of the sphere when the pressure reaches a value $p = 4tY_0 / R$

The strains are related to the radial displacements by $\varepsilon_{\theta\theta} = u / R$. Therefore

$$u = Y_0 R\left(\frac{2(1-\nu)}{E} + \frac{1}{2h}\right)$$

5. The figure shows a beam that is clamped at one end and pinned at the other. The beam has area moments of inertia $I_{22} = I_{11} = I, \; I_{12} = 0$. Calculate the buckling load (use the buckling mode that gives the lowest load).



We can follow the procedure from class. For the inertia matrix given, and since there is no transverse force or axial force per unit length, the governing equations for the transverse deflection and the axial force are

$$EI\frac{d^4u_1}{dx_3^4} = T_3\frac{d^2u_1}{dx_3^2} \qquad \frac{dT_3}{dx_3} = 0$$

We know that $T_3 = -P$ at $x_3 = L$, so we see that $T_3 = -P$ everywhere along the length of the beam.

The equation for transverse motion becomes

$$EI\frac{d^4u_1}{dx_3^4} + P\frac{d^2u_1}{dx_3^2} = 0$$

We try a general solution of the form

$$u_1 = \begin{bmatrix} \sin kx_3 & \cos kx_3 & x_3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

Where A,B,C,D and *k* are constants (as always the wave number $k = 2\pi/\lambda$ where λ is the wavelength of the buckling mode). Substituting into the governing equation gives

$$\left(EIk^4 - Pk^2\right)\begin{bmatrix} \sin kx_3 & \cos kx_3 & x_3 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

So $P = EIk^2$. The boundary conditions at the ends of the beam:

(i) At $x_3 = 0$ the slope and displacement are zero $u_1 = du_1/dx_3 = 0$

(ii) At $x_3 = L$ the displacement and moment are both zero, so $u_1 = EId^2u_1/dx_3^2 = 0$.

We can write the four boundary conditions in matrix form

$$\left[H\right]\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

(MATLAB can do all the tedious derivatives and substitutions)

```
syms k L x3 u real
syms
u = [sin(k*x3),cos(k*x3),x3,1]
H = [subs(u,x3,0);...
     subs(diff(u,x3),x3,0);...
     subs(u,x3,L);...
     subs(diff(u,x3,2),x3,L)]
 simplify(det(H))
```

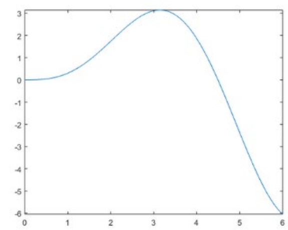
H =

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ k & 0 & 1 & 0 \\ \sin(L\,k) & \cos(L\,k) & L & 1 \\ -k^2\sin(L\,k) & -k^2\cos(L\,k) & 0 & 0 \end{pmatrix}$$

ans = $k^2\,(\sin(L\,k) - L\,k\cos(L\,k))$

It follows that

$$\left(\sin(Lk) - Lk\cos(Lk)\right) = 0$$



This has to be solved numerically. We have to be a bit careful because $Lk = 0$ is a solution and the MATLAB 'fsolve' will return something close to zero if given a bad initial guess. But if we plot the the function we see it has a root near *Lk=5* .

```
fsolve(@(lk) sin(lk)/lk-cos(lk),5)
```

gives Lk=4.4934.

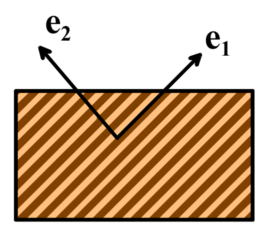
The buckling load follows as $P_{crit} = 4.4934^2 \dfrac{EI}{L^2}$

6. The figure shows a fiber reinforced composite laminate.

(i) When loaded in uniaxial tension parallel to the fibers, it fails at a stress of 500MPa.

(ii) When loaded in uniaxial tension transverse to the fibers, it fails at a stress of 250 MPa.

(iii) When loaded at 45 degrees to the fibers, it fails at a stress of 223.6 MPa



Failure in the laminate is to be predicted using the Tsai-Hill criterion

$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}^2} + \frac{\sigma_{12}^2}{\sigma_{SS}^2} = 1$$

6.1 Use the measurements to calculate values for the parameters $\sigma_{TS1}, \sigma_{TS2}, \sigma_{SS}$.

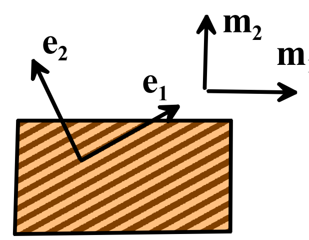
1. If the laminate is loaded in uniaxial tension parallel to the fibers, the material fails when $\sigma_{11} = \sigma_{TS1}$. It follows that $\sigma_{TS1} = 500MPa$
2. If the laminate is loaded in uniaxial tension perpendicular to the fibers. The material fails when $\sigma_{22} = \sigma_{TS2}$. It follows that $\sigma_{TS2} = 250MPa$
3. If the laminate in uniaxial tension with stress σ_0 at 45 degrees to the fibers (horizontally in the figure), we can use the basis change formulas to show that the stresses in the basis aligned parallel and perpendicular to the fibers $\sigma_{11} = \sigma_{22} = \sigma_{12} = \sigma_0/2$. (See problem 2 from HW4 2018 for details of this calculation). Substituting these into the failure criterion then shows that at failure

$$\left(\frac{\sigma_0/2}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_0/2}{\sigma_{TS2}}\right)^2 - \frac{(\sigma_0/2)^2}{\sigma_{TS1}^2} + \frac{(\sigma_0/2)^2}{\sigma_{SS}^2} = 1$$

We can solve this for σ_{SS}

$$\sigma_{SS} = \sigma_{TS2}\sigma_0 / \sqrt{4\sigma_{TS2}^2 - \sigma_0^2} = \frac{250 \times 223.6}{\sqrt{4 \times 250^2 - 223.6^2}} = 125MPa\,.$$

6.2 The laminate is then loaded in uniaxial tension at 30 degrees to the fibers. Calculate the expected failure stress under this loading, assuming that the material can be characterized using the Tsai-Hill failure criterion.

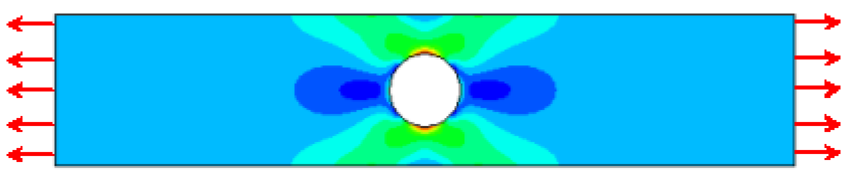


We have to use the basis change formulas to find the stress components in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis, then substitute the stress components into the failure criterion.

$$\begin{bmatrix} \sigma_{11}^{\mathbf{e}} & \sigma_{12}^{\mathbf{e}} \\ \sigma_{12}^{\mathbf{e}} & \sigma_{22}^{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sigma_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} = \frac{\sigma}{4} \begin{bmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

$$\Rightarrow \frac{\sigma^2}{16}\left[\left(\frac{3}{500}\right)^2 + \left(\frac{1}{250}\right)^2 - \frac{3}{250^2} + \frac{3}{125^2}\right] = 1$$

$$\Rightarrow \sigma = 285.7MPa$$



7. A specimen of steel has a yield stress of 500MPa. Under fully reversed cyclic loading at a stress amplitude of 200 MPa it is found to fail after 10^4 cycles, while at a stress amplitude of 100MPa it fails after 10^5 cycles. This material is to be used to fabricate a plate, with thickness h, containing circular holes with radius $a<<h$. The plate will be subjected to constant amplitude fully reversed cyclic uniaxial stress far from the holes, and must have a life of at least 10^5 cycles. What is the maximum stress amplitude (far from the hole) that the plate can withstand?

The stress is below the yield stress, so the material will fail by high cycle fatigue. We can estimate the number of cycles to failure using Basquin's law

$$\Delta\sigma N_f^b = C$$

We also know that the plate with a hole has a stress concentration factor of 3 (i.e the stress near the hole is three times the stress far from the hole - see homework 5, problem 4)

To survive 10^5 cycles the stress amplitude near the hole cannot exceed 100MPa, which means that the stress far from the hole cannot exceed 100/3 MPa

8. A spherical pressure vessel with internal radius *a* and external radius *b*=1.5*a* is repeatedly pressurized from zero internal pressure to a maximum value p. The sphere has yield stress *Y,* ultimate tensile strength σ_{UTS} and its fatigue behavior (under fully reversed uniaxial tension) can be characterized by Basquin's law $\sigma_a N^b = C$. You can assume that the elastic stresses in the vessel are given by

$$\sigma_{rr} = -p\frac{a^3(b^3 - r^3)}{r^3(b^3 - a^3)} \qquad \sigma_{\theta\theta} = p\frac{a^3(b^3 + 2r^3)}{2r^3(b^3 - a^3)}$$

8.1 Find an expression for the fatigue life of the vessel in terms of p, and relevant geometric and material properties. Assume that the effects of mean stress can be approximated using Goodman's rule. Assume that $p/Y < 2(1 - a^3/b^3)/3$

The maximum tensile stress occurs at the inner wall $r = a$, which gives

$$\sigma_{\theta\theta} = p\frac{(b^3 + 2a^3)}{2(b^3 - a^3)} = 1.1316p$$

The fatigue life satisfies the equation

$$1.1316pN^b = C\left(1 - \frac{1.1316p}{\sigma_{UTS}}\right)$$

The life is therefore

$$N = \left\{C\left(\frac{1}{1.1316p} - \frac{1}{\sigma_{UTS}}\right)\right\}^{1/b}$$