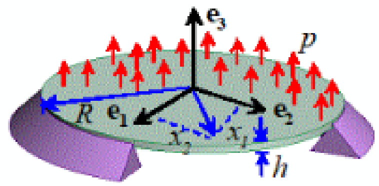
Review: Plates

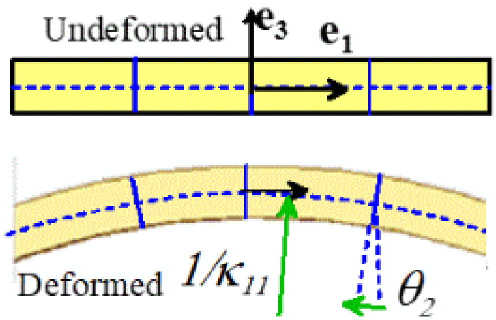
Goal: Calculate (1) Displacement of mid-plane $\mathbf{u}(x_1,x_x)=u_1\mathbf{e}_1+u_2\mathbf{e}_2+u_3\mathbf{e}_3$

(2) Rotation of x-section $\boldsymbol{\theta}(x_1,x_2)=\theta_1\mathbf{e}_1+\theta_2\mathbf{e}_2$

(3) Curvature tensor $\boldsymbol{\kappa}(x_1,x_2)$



Deformation:



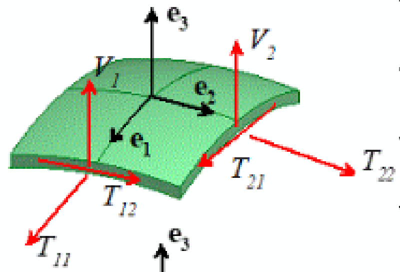
Kirchhoff theory (no shear): $\theta_1=\partial u_3/\partial x_2 \qquad \theta_2=-\partial u_3/\partial x_1$

(Mindlin theory allows x-sect to rotate relative to neutral axis)

Curvature $\begin{bmatrix}\kappa_{11} & \kappa_{12}\\ \kappa_{12} & \kappa_{22}\end{bmatrix} \quad \kappa_{11}=-\dfrac{\partial^2 u_3}{\partial x_1^2} \quad \kappa_{22}=-\dfrac{\partial^2 u_3}{\partial x_2^2} \quad \kappa_{12}=-\dfrac{\partial^2 u_3}{\partial x_1\partial x_2}$

Strains $\varepsilon_{11}=\dfrac{\partial u_1}{\partial x_1}+\kappa_{11}x_3 \qquad \varepsilon_{22}=\dfrac{\partial u_2}{\partial x_2}+\kappa_{22}x_3 \qquad \varepsilon_{12}=\dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2}+\dfrac{\partial u_2}{\partial x_1}\right)+\kappa_{12}x_3$

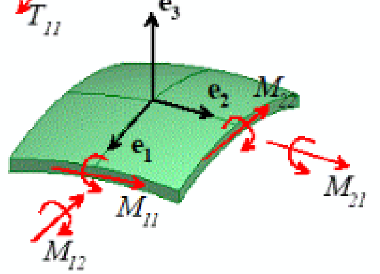
(Mindlin theory has additional shear strains)



Stresses (assume plane stress)

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix}$$

Internal Forces and moments



$$\begin{bmatrix}T_{11}\\ T_{22}\\ T_{12}\end{bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}dx_3 \qquad \begin{bmatrix}M_{11}\\ M_{22}\\ M_{12}\end{bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}x_3dx_3$$

V_1, V_2 are constraint forces

Review: Plates

Force – displacement relation:

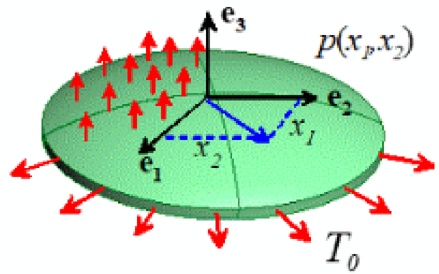
$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix} = \frac{Eh}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \partial u_1 / \partial x_1 \\ \partial u_2 / \partial x_2 \\ (\partial u_1 / \partial x_2 + \partial u_2 / \partial x_1)/2 \end{bmatrix}$$

Moment-curvature relation:

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu) \end{bmatrix} \begin{bmatrix} \kappa_{11} \\ \kappa_{22} \\ \kappa_{12} \end{bmatrix}$$

Equations of motion:

Assume (1) Only transverse loading
(2) Uniform force per unit length around perimeter
(ABAQUS can handle more general loading)



In-plane loading $T_{11} = T_{22} = T_0 \qquad T_{12} = 0$

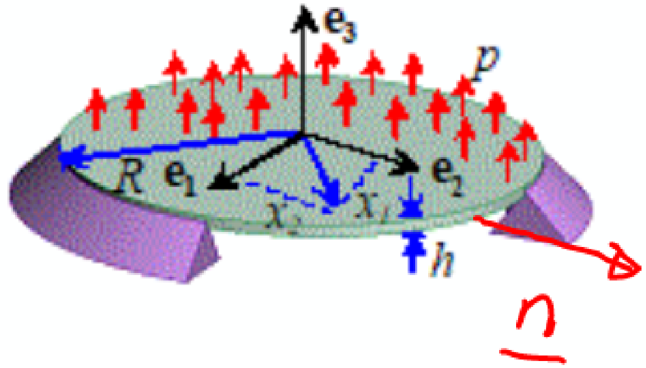
Linear Momentum **F**=ma $\dfrac{\partial V_1}{\partial x_1} + \dfrac{\partial V_2}{\partial x_2} + p_3 = \rho h a_3$

Angular Momentum (neglect rotational inertia)

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - V_1 + T_0 \frac{\partial u_3}{\partial x_1} = 0$$

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - V_2 + T_0 \frac{\partial u_3}{\partial x_2} = 0$$

Boundary Conditions



Clamped edge

$$u_3 = 0$$

$$\frac{\partial u_3}{\partial x_1} n_1 + \frac{\partial u_3}{\partial x_2} n_2 = 0$$

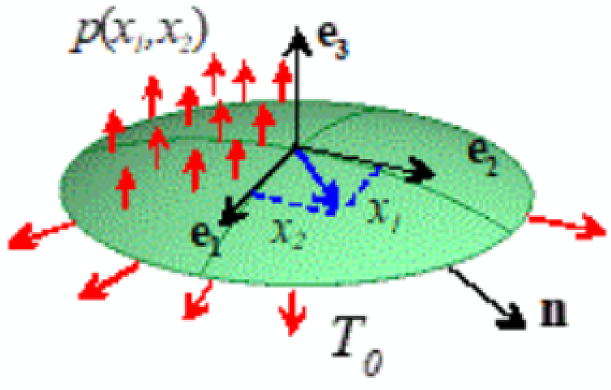
Pinned edge

$$u_3 = 0$$

$$[n_1 \; n_2] \begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = 0$$

Free edge: Complicated !

Simplified Special Cases: Membrane



Neglect bending resistance $\frac{Eh^3}{T_0 R^2} << 1$

$M_{11} = M_{22} = M_{12} = 0$

Angular momentum: $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = T_0 \begin{bmatrix} \partial u_3 / \partial x_1 \\ \partial u_3 / \partial x_2 \end{bmatrix}$

Linear momentum $\partial V_1 / \partial x_1 + \partial V_2 / \partial x_2 + p_3 = \rho h a_3$

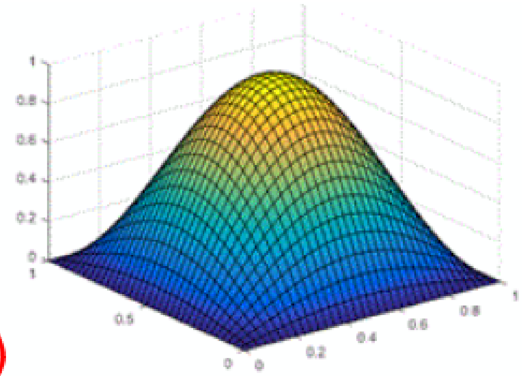
Combine $T_0 \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} \right) + p_3 = \rho h a_3$

$$T_0 \nabla^2 u_3 + p_3 = \rho h a_3$$

Boundary conditions $u_3 = 0$ (pinned or clamped)

Example: A square membrane with fixed edges is subjected to a sinusoidal pressure $p = p_0\{\sin(\pi x_1 / a)\sin(\pi x_2 / a)\}$

Find the deflection



Governing eq $T_0 \nabla^2 u_3 = -p_3$ (static) (1)

Boundary Condition $u_3 = 0$ $x_1 = 0$ $x_1 = a$

$x_2 = 0$ $x_2 = a$

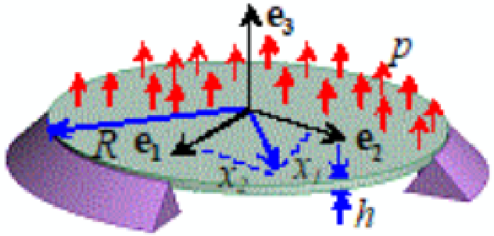
Guess $u_3 = U_0 \sin \pi x_1 / a \quad \sin \pi x_2 / a$

(1) $\Rightarrow$ $T_0 U_0 \left(-\frac{\pi^2}{a^2} - \frac{\pi^2}{a^2} \right) \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a} = -p_0 \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}$

$\Rightarrow U_0 = \frac{p_0 a^2}{2\pi^2 T_0} \Rightarrow \boxed{u_3 = \frac{p_0 a^2}{2\pi^2 T_0} \sin \frac{\pi x_1}{a} \sin \frac{\pi x_2}{a}}$

Simplified cases: Plate with no in-plane loading

Equations with $T_0 = 0$ reduce to



Moment-curvature

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} -\partial^2 u_3 / \partial x_1^2 \\ -\partial^2 u_3 / \partial x_2^2 \\ -\partial^2 u_3 / \partial x_1 \partial x_2 \end{bmatrix}$$

Angular momentum

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \partial M_{11}/\partial x_1 + \partial M_{12}/\partial x_2 \\ \partial M_{12}/\partial x_1 + \partial M_{22}/\partial x_2 \end{bmatrix}$$

Linear momentum

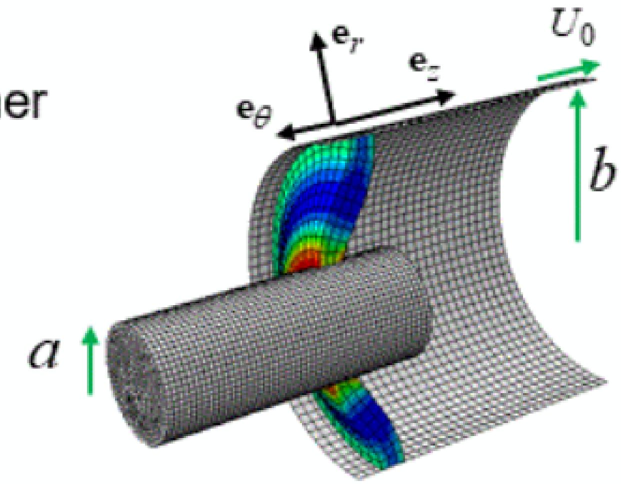
$$\partial V_1/\partial x_1 + \partial V_2/\partial x_2 + p_3 = \rho h a_3$$

Combine:

$$\frac{Eh^3}{12(1-\nu^2)} \left\{ \frac{\partial^4 u_3}{\partial x_1^4} + 2\frac{\partial^4 u_3}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u_3}{\partial x_2^4} \right\} - p_3 + \rho h a_3 = 0$$

or $\frac{Eh^3}{12(1-\nu^2)} \nabla^4 u_3 - p_3 + \rho h a_3 = 0$

Example: A circular annular plate with internal radius a and external radius b is clamped at its inner and outer edges. The inner edge is held fixed, and the outer edge is translated parallel to the $\mathbf{e}_z$ by a distance U_0. Find the displacement field in the plate



Governing eq: $\frac{Eh^3}{12(1-\nu^2)} \nabla^4 u_z = 0$

Boundary conditions $u_z = 0 \quad \frac{\partial u_z}{\partial r} = 0 \quad r = a$

$u_z = U_0 \quad \frac{\partial u_z}{\partial r} = 0 \quad r = b$

Biharmonic eq $\nabla^4 u_z = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta} \right)^2 u_z = 0$

Guess $u_z = u_z(r)$

```
syms r a b U0 xi real
syms uz(r)
a = 1; b = 3; U0 = 1;
laplacian = diff(uz(r),r,2) + diff(uz(r),r)/r;
biharmonic = simplify(diff(laplacian,r,2) + diff(laplacian,r)/r);
BC1 = subs(uz(r),r,a) ==0;
BC2 = subs(diff(uz(r),r),r,a) ==0;
BC3 = subs(uz(r),r,b) ==U0;
BC4 = subs(diff(uz(r),r),r,b) ==0;
ur = simplify(dsolve(biharmonic==0,BC1,BC2,BC3,BC4))
fplot(ur,[a,b],'Color',[1 0 0],'LineWidth',2)
axes1 = gca;
ylabel('u_z/U_0');
xlabel('r/a');
box(axes1,'on');
axis(axes1,'tight');
set(axes1,'FontSize',14,'XGrid','on','YGrid','on');
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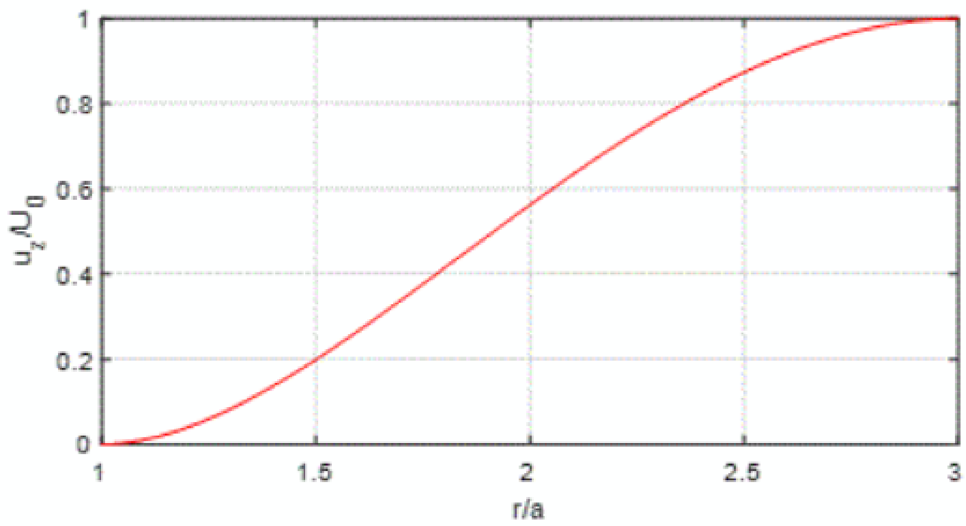


Warning: MATLAB gives wrong solution to

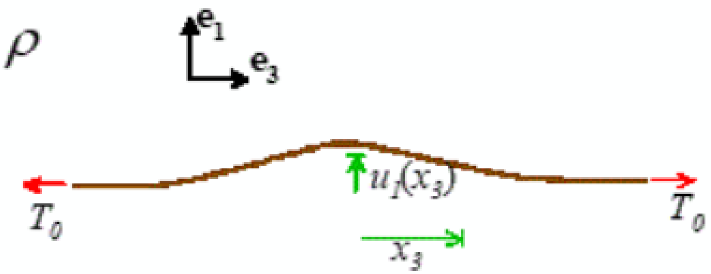
$D^4 u_z = p$ with $p \neq 0$!

12 Dynamics

Goal: Understand wave propagation & vibrations in elastic solids

12.1 Traveling wave solutions in strings

Example: An infinite string with x-sect area *A* and mass density ρ is stretched by tension T_0
At time *t=0* it is at rest, and has transverse deflection $w_0(x_3)$
Find the subsequent motion



Governing Equation $T_0 \dfrac{\partial^2 u_1}{\partial x_3^2} = \rho A \dfrac{\partial^2 u_1}{\partial t^2}$ Wave eq !

$$\Rightarrow \frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_3}{\partial t^2} \qquad c = \sqrt{\frac{T_0}{\rho A}} \quad \text{(Wave speed)}$$

Solution $u_1 = f(x_3 - ct) + g(x_3 + ct)$ for any f, g

Note $\frac{\partial^2 f}{\partial x_3^2} = f''$ $\quad \frac{\partial^2 f}{\partial t^2} = c^2 f''$ $\qquad f'(\lambda) = \frac{\partial f}{\partial \lambda}$

Find f, g from initial conditions

$u_1 = W_0(x_3)$ @ $t=0$ $\Rightarrow$ $f(x_3) + g(x_3) = W_0(x_3)$ (1)

$\partial u_1/\partial t = 0$ $\quad t=0$ $\quad -\cancel{c}f'(x_3) + \cancel{c}g'(x_3) = 0$

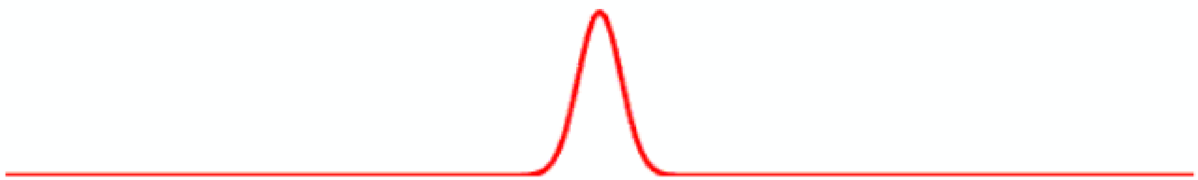
Integrate $\Rightarrow$ $-f + g = A$ ← const! (2)

Solve (1) & (2) $\Rightarrow$ $g(x_3) = [W_0(x_3) + A]/2$

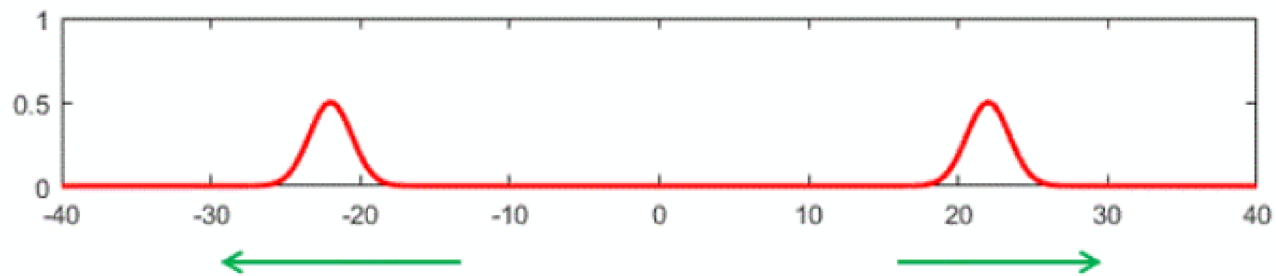
$f(x_3) = [W_0(x_3) - A]/2$

$$u_1(x_3, t) = [\underbrace{W_0(x_3 - ct)}_{\text{wave propagating in } +x_3 \text{ dir}} + \underbrace{W_0(x_3 + ct)}_{-x_3 \text{ dir}}]/2$$

Solution to wave equation



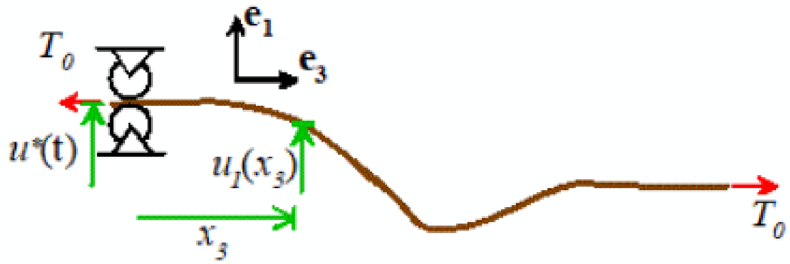
Initial condition $w_0 = \exp(-x_3^2 / 4)$



$$g(t, x_3) = \frac{1}{2}\exp(-(x_3 + ct)^2 / 4) \qquad f(t, x_3) = \frac{1}{2}\exp(-(x_3 - ct)^2 / 4)$$

Travelling waves, propagate at speed *c*

Example: An semi- infinite string with x-sect area A and mass density ρ is stretched by tension T_0
At time $t=0$ it is at rest, with zero transverse deflection



The end at $x_3 = 0$ has prescribed displacement $u^*(t)$

Find the subsequent motion

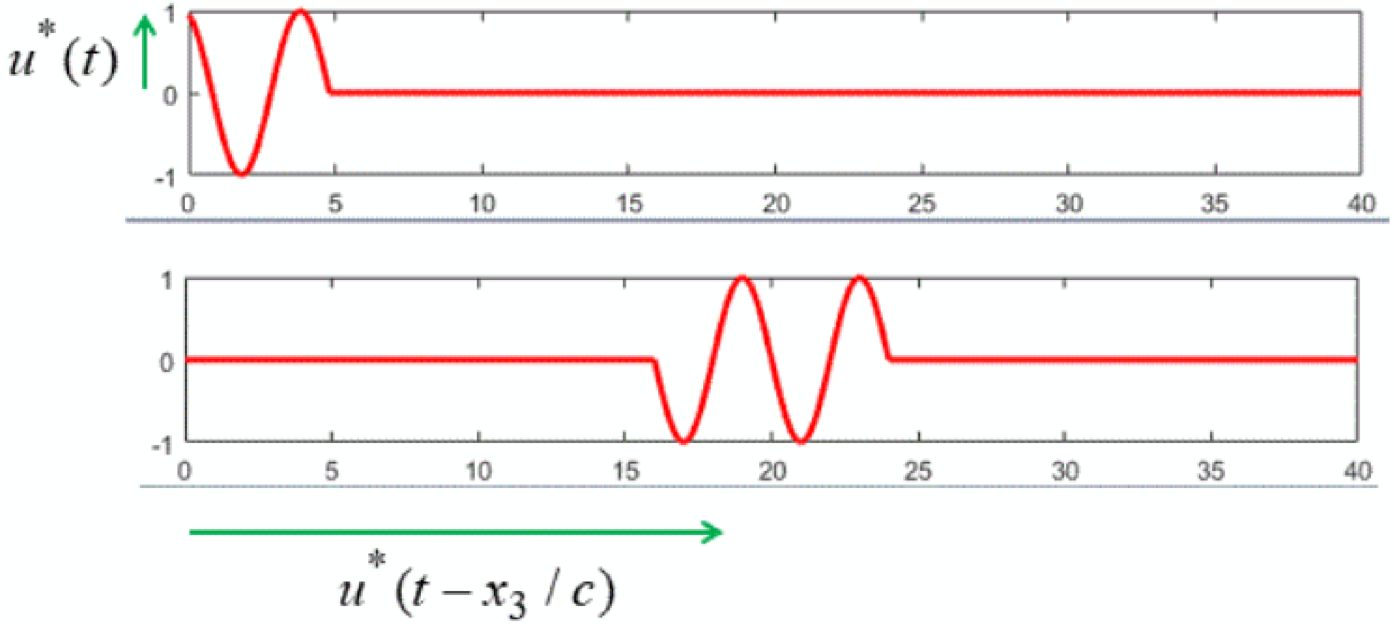
Expect solution to be a wave propagating in x_3 dir

Wave travels @ speed c

Disturbance $u^*(t)$ @ $x_3 = 0$ will occur at time $t = x_3/c$ @ x_3

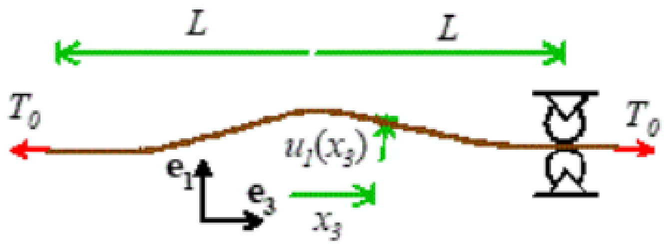
Hence $$u_1(t, x_3) = \begin{cases} u^*(t - x_3/c) & t > x_3/c \\ 0 & t < x_3/c \end{cases}$$

Semi-infinite string forced at one end



Wave arrives at x_3 a time x_3 / c later

Example: A string with length *2L* x-sect area *A* and mass density is stretched by tension T_0
At time *t=0* it is at rest, and has transverse deflection $w_0(x_3)$
It is fixed at $x_3 = L$ and free at $x_3 = -L$
Find the subsequent motion



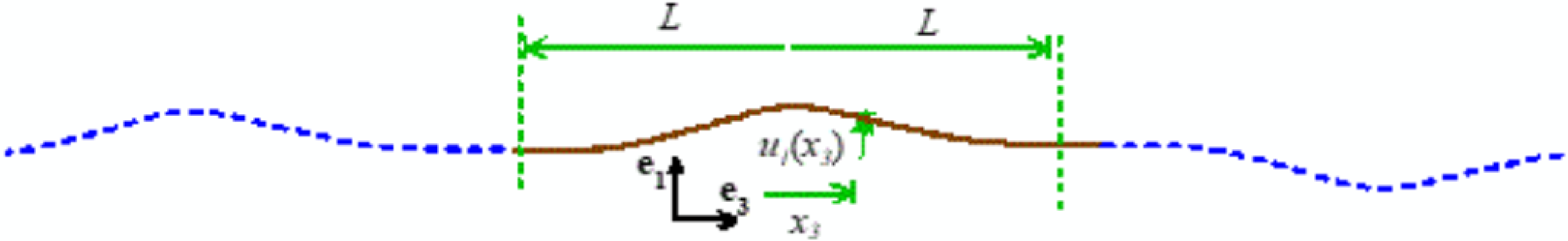
Boundary conditions : $u_1 = 0$ @ $x_3 = L$

$$T_1 = T_0 \frac{\partial u_1}{\partial x_3} = 0 \quad @ \; x_3 = -L$$

Solve by superposition :

(1) Extend string to +/- ∞
(2) Introduce initial conditions in regions $|x_3| > L$ that will satisfy BCs

(3) Use travelling wave sol for ∞ string



To be continued