# Course Outline

1. Fundamental Postulates of Solid Mechanics
2. Introduction to FEA using ABAQUS
3. Math Review, introduction to tensors and index notation
4. Describing Deformations
5. Describing Forces
6. Equations of Motion
7. Linear Elastic Stress-Strain Relations
8. Analytical Solutions for Linear Elastic Solids
9. Energy Methods for Linear Elastic Solids
10. Implementing the Finite Element Method for Elastic Solids
11. Solids with special shapes – beams and plates
12. Dynamic elasticity – waves and vibrations
13. Plasticity
14. Modeling failure

**Exam Topics**

# Concept Checklist

**2. FEA analysis**

- Be able to idealize a solid component as a 3D continuum, rod, shell or plate
- Understand how to choose a material model for a component or structure
- Be familiar with features of a finite element mesh; be able to design an suitable mesh for a component
- Understand the role of the FE mesh as a way to interpolate displacement fields
- Understand the difference between solid, shell and beam elements
- Understand that selecting inappropriate element types and poor mesh design may lead to inaccurate results
- Understand how to select boundary conditions and loading applied to a mesh
- Understand that for static analysis boundary conditions must prevent rigid motion to ensure that FEA will converge
- Understand use of tie constraints to bond meshes or to bond a rigid surface to a part
- Be able to analyze contact between deformable solids
- Be able to choose a static, explicit dynamic, or implicit dynamic analysis;

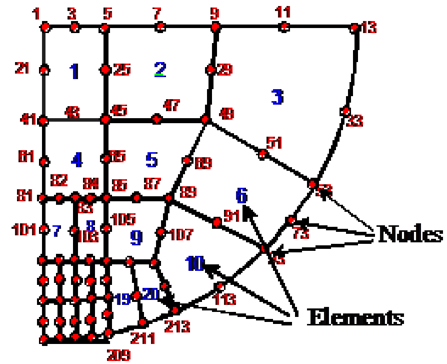
- Be able to interpret and draw conclusions from analysis predictions; have the physical insight to recognize incorrect predictions
- Be able to use dimensional analysis to simplify finite element simulations

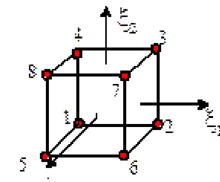
# FEA Analysis

**Features of FE Mesh**

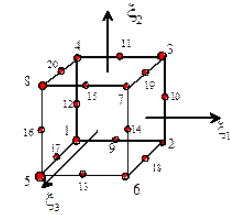
**Nodes:** Used to track motion of points in solid
**Elements:** Main purpose is to interpolate displacements between values at nodes.

ABAQUS offers linear (nodes at corners) and quadratic (nodes at mid-sides) elements



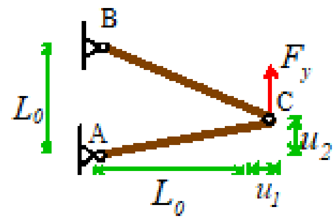
Linear



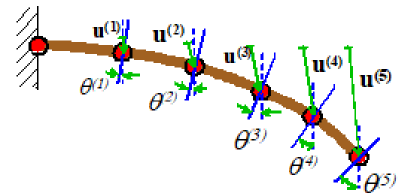
Quadratic

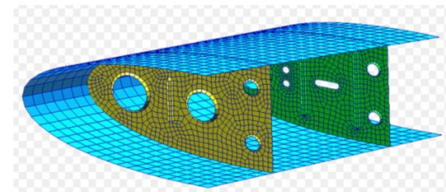
**Special element types**

**Truss:** Special displacement interpolation for 2 force members
**Beam:** Special displacement interpolation for slender member. Have rotation DOFS/moments
**Plate/Shell**: Special displacement interpolation for thin sheets that can deform out-of-plane. Rotations/moments



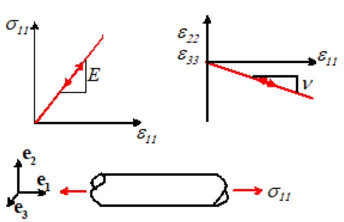

Deformed



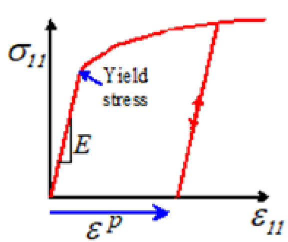



**Materials** (Some examples)

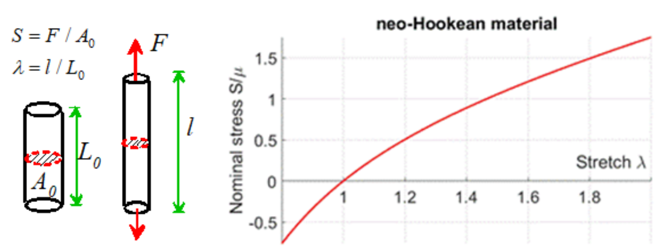
**Linear Elasticity:** OK for most materials subjected to small loads
**Plasticity** Metals beyond yield
**Hyperelasticity**: Large strain reversible model used for rubbers
**Viscoelasticity:** Time dependent material used for polymers/tissue



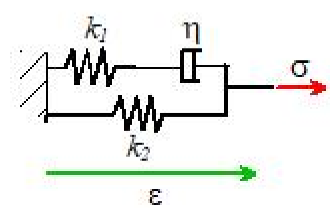

Linear Elastic




Elastic-Plastic




Hyperelastic



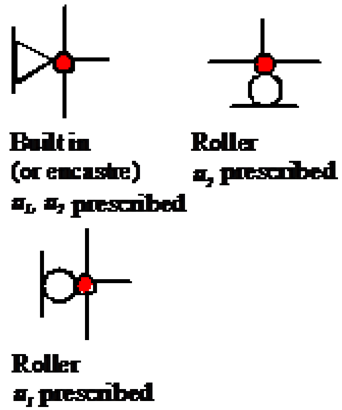

Visco-elastic

# FEA Analysis

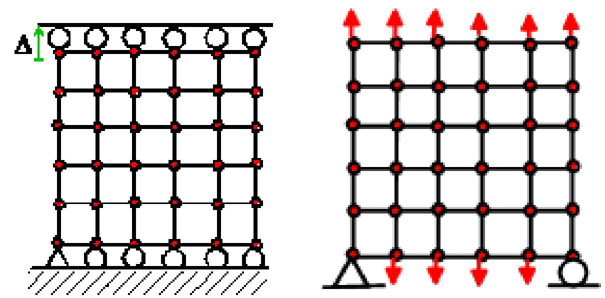
## Boundary Conditions

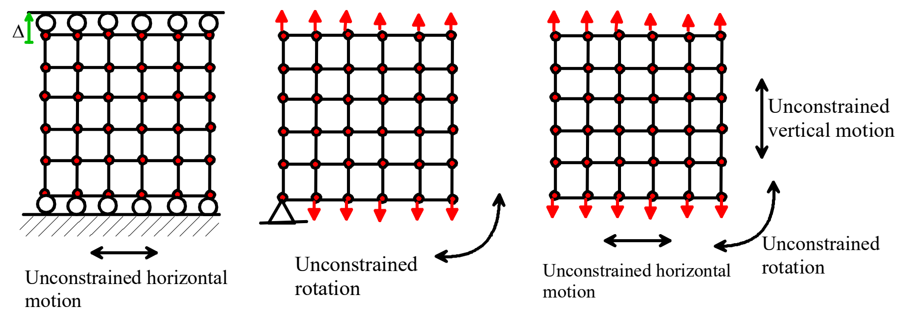
We can apply

1. Prescribed displacements
2. Forces on nodes
3. Pressure on element faces
4. Body forces
5. For some elements, can apply rotations/moments




*Properly constrained solids*





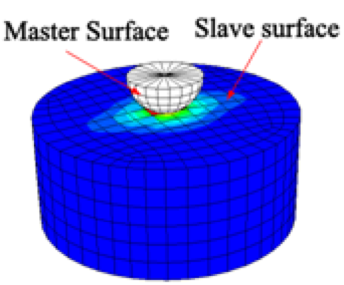

*Incorrectly constrained solids*

For static analysis we have to make sure we stop solid from translating/rotating

## Contact

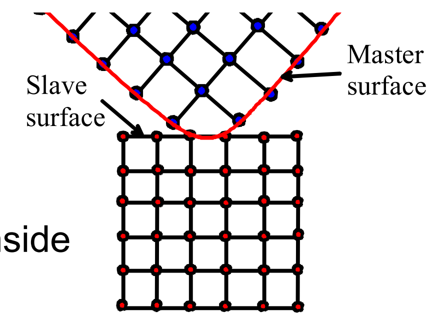
Select

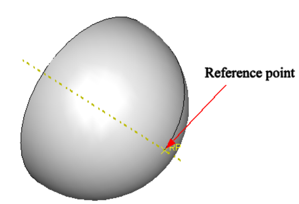
1. Contact algorithm (Surface/Node Based)
2. Constitutive law for contact
   - "Soft" or "Hard" normal contact
   - Friction law




### Master/slave pairs

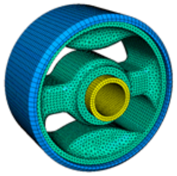
Nodes on slave surface are prevented from penetrating inside master surface

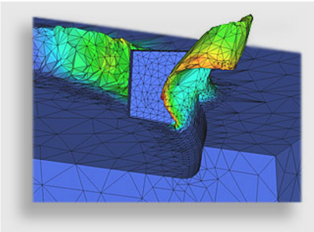


# FEA Analysis

## Solution Procedures





Small strain –v- large strain (NLGEOM)

Static Linear analysis solves $\mathbf{Ku} = \mathbf{r} \quad \mathbf{u} = \mathbf{u}^*$

Nonlinear problem: solves $\mathbf{R}(\mathbf{u}) = \mathbf{F}^* \quad \mathbf{u} = \mathbf{u}^*$ using Newton-Raphson iteration

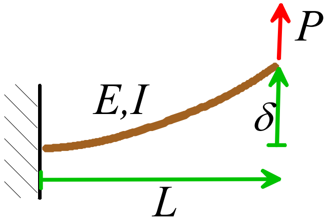
Explicit Dynamics: solves $\mathbf{M\ddot{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^* \quad \mathbf{u} = \mathbf{u}^*$ using 2nd order forward Euler scheme

Implicit Dynamics: solves $\mathbf{M\ddot{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^* \quad \mathbf{u} = \mathbf{u}^*$ using 2nd order backward Euler scheme

Special procedures: modal dynamics, buckling ('Linear Perturbation steps')

## Using Dimensional Analysis

Input data to code $\delta = f(P, E, I, L)$




Dimensionless form (others are possible) $\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$

If we know $\delta = f(P, EI, L)$

Then $\frac{\delta}{L} = f\left(\frac{PL^2}{EI}\right)$

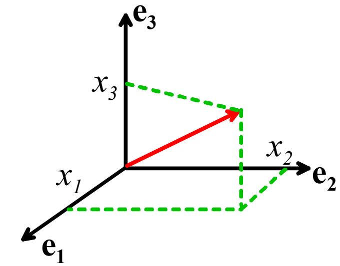
If we know problem is linear, then $\frac{\delta}{L} = C\frac{PL^2}{EI}$ for some constant C

# Concept Checklist

**3. Mathematics**

- Understand the concepts of scalar, vector and tensor fields; understand use of Cartesian and polar basis vectors to represent vector and tensor fields
- Be able to compute gradient and divergence of scalar, and vector fields in Cartesian and Polar coordinates;
- Understand the concept of a tensor as a linear mapping of vectors;
- Be able to create a tensor using vector dyadic products; be able to add, subtract, multiply tensors; be able to calculate contracted products of tensors; be able to find the determinant, eigenvalues and eigenvectors of tensors; understand the spectral decomposition of a symmetric tensor;
- Be familiar with special tensors (identity, symmetric, skew, and orthogonal)
- Be able to transform tensor components from one basis to another.
- Be familiar with the conventions of index notation and perform simple algebra with index notation
- Be able to calculate the divergence of a symmetric tensor field in Cartesian or polar coordinates (eg to check the stress equilibrium equation)

# Math




Position $\mathbf{r} = x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$

Scalar Field $\phi(x_i)$ gradient $\nabla \phi = \frac{\partial \phi}{\partial x_i} \mathbf{e}_i$

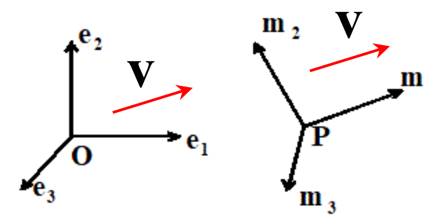
Vector Field $\mathbf{v}(x_i)$ gradient $\nabla \mathbf{v} = \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$

Tensor: linear map of vectors onto vectors $\mathbf{v} = \mathbf{S} \cdot \mathbf{u} \equiv v_i = S_{ij} u_j$ $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$

Dyadic product of vectors $\mathbf{S} = (\mathbf{a} \otimes \mathbf{b})$ $\mathbf{S} \cdot \mathbf{u} = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\mathbf{a}$ $S_{ij} = a_i b_j$

General tensor as a sum of dyads $\mathbf{S} = S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$

Basis change formulas




Vectors: $\mathbf{v} = v_i^{(\mathbf{m})} \mathbf{m}_i = v_i^{(\mathbf{e})} \mathbf{m}_i$ $v_i^{(\mathbf{m})} = Q_{ij} v_j^{(\mathbf{e})}$

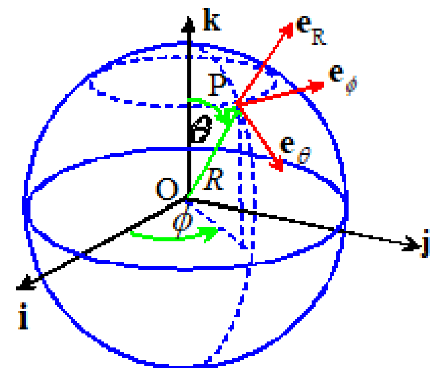
Tensors: $\mathbf{S} = S_{ij}^{(\mathbf{m})} \mathbf{m}_i \otimes \mathbf{m}_j = S_{ij}^{(\mathbf{e})} \mathbf{e}_i \otimes \mathbf{e}_j$

$S_{kl}^{(\mathbf{m})} = Q_{ki} S_{ij}^{(\mathbf{e})} Q_{lj}$ $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ $[Q] = \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 & \mathbf{m}_1 \cdot \mathbf{e}_3 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 & \mathbf{m}_2 \cdot \mathbf{e}_3 \\ \mathbf{m}_3 \cdot \mathbf{e}_1 & \mathbf{m}_3 \cdot \mathbf{e}_2 & \mathbf{m}_3 \cdot \mathbf{e}_3 \end{bmatrix}$

# Gradients in Polar Coordinates

Position $\mathbf{r} = R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k}$

Vector $\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi$



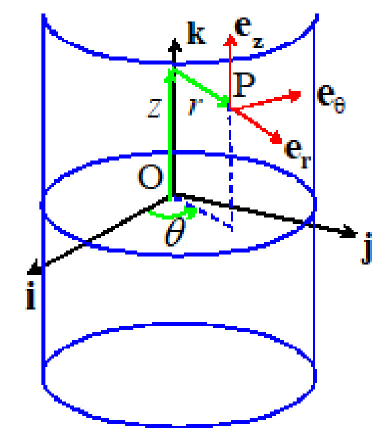

Gradient of a scalar $$\nabla f = \mathbf{e}_R\frac{\partial f}{\partial R} + \mathbf{e}_\theta\frac{1}{R}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{R\sin\theta}\frac{\partial f}{\partial \phi}$$

Gradient of a vector $$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R} & \frac{1}{R\sin\theta}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R} \\ \frac{\partial v_\theta}{\partial R} & \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R} & \frac{1}{R\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \cot\theta\frac{v_\phi}{R} \\ \frac{\partial v_\phi}{\partial R} & \frac{1}{R}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta\frac{v_\theta}{R} + \frac{v_R}{R} \end{bmatrix}$$

Divergence of a vector $$\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_R}{\partial R} + \frac{2v_R}{R} + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \cot\theta\frac{v_\theta}{R} + \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Position $\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$

Vector $\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z$




Gradient of a scalar $$\nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z\frac{\partial f}{\partial z}$$

Gradient of a vector $$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector $$\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$$

# Tensor Operations

Operations on 3x3 matrices also apply to tensors

Addition $\mathbf{U}=\mathbf{S}+\mathbf{T}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} S_{11}+T_{11} & S_{12}+T_{12} & S_{13}+T_{13} \\ S_{21}+T_{21} & S_{22}+T_{22} & S_{23}+T_{23} \\ S_{31}+T_{31} & S_{32}+T_{32} & S_{33}+T_{33} \end{bmatrix}$$

Vector/Tensor product $\mathbf{v}=\mathbf{S}\cdot\mathbf{u}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} S_{11}u_1+S_{12}u_2+S_{13}u_3 \\ S_{21}u_1+S_{22}u_2+S_{23}u_3 \\ S_{31}u_1+S_{32}u_2+S_{33}u_3 \end{bmatrix}$$

$\mathbf{v}=\mathbf{u}\cdot\mathbf{S}$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} u_1S_{11}+u_2S_{21}+u_3S_{31} \\ u_1S_{12}+u_2S_{22}+u_3S_{32} \\ u_1S_{13}+u_2S_{23}+u_3S_{33} \end{bmatrix}$$

Tensor product $\mathbf{U}=\mathbf{T}\cdot\mathbf{S}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}S_{11}+T_{12}S_{21}+T_{13}S_{31} & T_{11}S_{12}+T_{12}S_{22}+T_{13}S_{32} & T_{11}S_{13}+T_{12}S_{23}+T_{13}S_{33} \\ T_{21}S_{11}+T_{22}S_{21}+T_{23}S_{31} & T_{21}S_{12}+T_{22}S_{22}+T_{23}S_{32} & T_{21}S_{12}+T_{22}S_{22}+T_{23}S_{32} \\ T_{31}S_{11}+T_{32}S_{21}+T_{33}S_{31} & T_{31}S_{12}+T_{32}S_{22}+T_{33}S_{32} & T_{31}S_{13}+T_{32}S_{23}+T_{33}S_{33} \end{bmatrix}$$

# Tensor Operations

Transpose

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^T = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u}$$

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Determinant $\det(\mathbf{S}) = S_{11}(S_{22}S_{33} - S_{23}S_{32}) - S_{22}(S_{12}S_{33} - S_{32}S_{13}) + S_{33}(S_{12}S_{23} - S_{22}S_{13})$

Eigenvalues/vectors $\mathbf{S} \cdot \mathbf{m} = \lambda \mathbf{m}$

$$\det(\mathbf{S} - \lambda \mathbf{I}) = 0$$

Spectral decomposition for symmetric **S**

$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i \mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)}$$

Inverse $\mathbf{S}^{-1} \cdot \mathbf{S} = \mathbf{I}$

Identity

$$\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric tensor $\mathbf{S} = \mathbf{S}^T$

Skew tensors $\mathbf{S}^T = -\mathbf{S}$

Proper orthogonal tensors $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ $\det(\mathbf{R}) = +1$

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

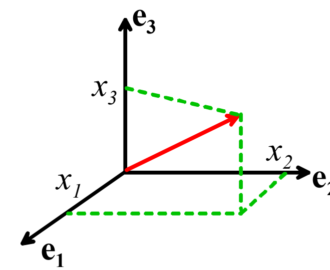
Inner product $\mathbf{S} : \mathbf{S} \equiv S_{ij}S_{ij} = S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + \ldots.$

Outer product $\mathbf{S} \cdot\cdot \mathbf{S} \equiv S_{ij}S_{ji} = S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + \ldots.$

# Index Notation Summary

Vector $\mathbf{x} = (x_1, x_2, x_3)$

Tensor $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$




Index Notation $\mathbf{x} \equiv x_i$ $\qquad \mathbf{S} \equiv S_{ij}$

Summation convention

$$\lambda = a_i b_i \equiv \lambda = \sum_{i=1}^{3} a_i b_i \equiv \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$$

$$c_i = S_{ik} x_k \equiv c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \begin{cases} c_1 = S_{11} x_1 + S_{12} x_2 + S_{13} x_3 \\ c_2 = S_{21} x_1 + S_{22} x_2 + S_{23} x_3 = \mathbf{Sx} \\ c_3 = S_{31} x_1 + S_{32} x_2 + S_{33} x_3 \end{cases}$$

$$\lambda = S_{ij} S_{ij} \equiv \lambda = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \equiv \lambda = S_{11} S_{11} + S_{12} S_{12} + \ldots + S_{31} S_{31} + S_{32} S_{32} + S_{33} S_{33} = \mathbf{S} : \mathbf{S}$$

$$C_{ij} = A_{ik} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \equiv \mathbf{C} = \mathbf{AB} \qquad C_{ij} = A_{ki} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \equiv \mathbf{C} = \mathbf{AB}^T$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad a_i \delta_{ij} = a_j \qquad \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

Permutation symbol

$$\in_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3; \quad 2,3,1 \quad \text{or} \quad 3,1,2 \\ -1 & i,j,k = 3,2,1; \quad 2,1,3 \quad \text{or} \quad 1,3,2 \\ 0 & \text{otherwise} \end{cases}$$

$$\in_{ijk} = \in_{kij} = \in_{jki} = -\in_{jik} = -\in_{kji} = -\in_{kji}$$

$$\in_{kki} = 0$$

$$\in_{ijk} \in_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{mk}$$

$$\lambda = \det \mathbf{A} \equiv \lambda = \frac{1}{6} \in_{ijk} \in_{lmn} A_{li} A_{mj} A_{nk} \qquad S_{ij}^{-1} = \frac{1}{2 \det(\mathbf{S})} \in_{ipq} \in_{jkl} S_{pk} S_{ql}$$

# Concept Checklist

**4. Deformations**

- Understand the concept and definition of a deformation gradient; be able to calculate a deformation gradient from a displacement field in Cartesian/polar coordinates; be able to calculate understand the physical significance of the Jacobian of the deformation gradient
- Understand Lagrange strain and its physical significance; be able to calculate Lagrange strain from deformation gradient tensor or displacement measurements.
- Know the definition of the infinitesimal strain tensor; understand that it is an approximate measure of deformation; be able to calculate infinitesimal strains from a displacement in Cartesian/polar coords
- Know and understand the significance of the compatibility equation for infinitesimal strain in 2D, and be able to integrate 2D infinitesimal strain fields to calculate a displacement field.
- Be able to calculate principal strains and understand their physical significance
- Be able to transform strain components from one basis to another

# Deformations

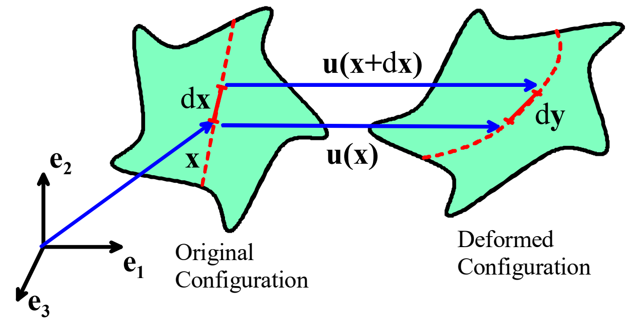
**Deformation Mapping:** $\mathbf{y}(\mathbf{x},t)$

**Displacement Vector:** $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{x}$

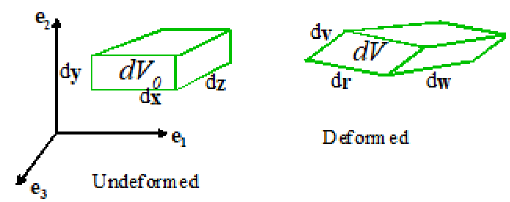
**Deformation Gradient:** $\mathbf{F} = \nabla \mathbf{y} = \nabla \mathbf{u} + \mathbf{I}$

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} + \delta_{ij}$$

$$d\mathbf{y} = \mathbf{F} d\mathbf{x} \qquad dy_i = F_{ij} dx_j$$



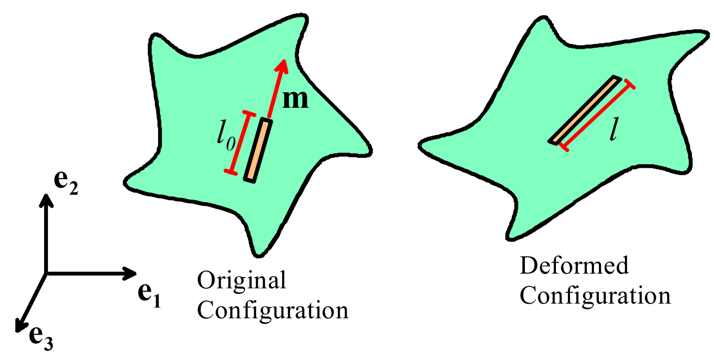

**Jacobian:** $J = \det(\mathbf{F}) \quad dV = J dV_0$




**Lagrange Strain:** $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})$

$$E_{ij} = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij})$$

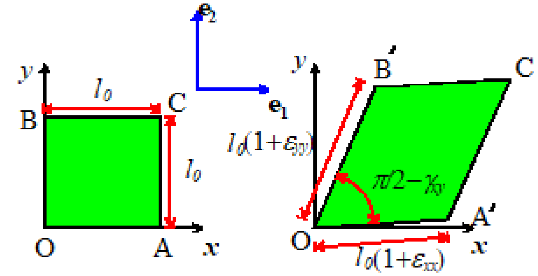
$$\frac{l^2 - l_0^2}{2 l_0^2} = \mathbf{m} \cdot \mathbf{E} \mathbf{m} = m_i E_{ij} m_j$$



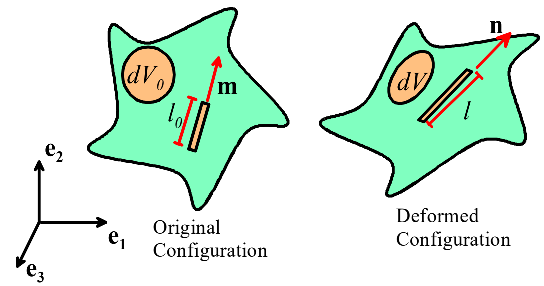

Can use this to find **E** given $l, l_0, \mathbf{m}$ for 3 (in 2D) directions

# Deformations

**Infinitesimal strain:** $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$



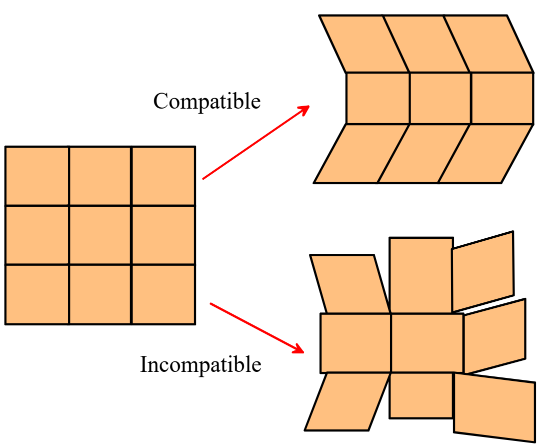

**Properties:**
(1) Approximate strain measure used only for small deformation
(2) For small strains $\boldsymbol{\varepsilon} \approx \mathbf{E}$
(3) Components quantify length and angle changes of unit cube
(4) $\mathbf{m} \cdot \boldsymbol{\varepsilon}\mathbf{m} = m_i \varepsilon_{ij} m_j \approx (l - l_0)/l_0$
(5) $trace(\boldsymbol{\varepsilon}) = \varepsilon_{kk} \approx (dV - dV_0)/dV_0$




## 2D Compatibility conditions

To be able to integrate strains (to find displacement)

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$




## Integrating strains

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \Rightarrow u_1 = \int \varepsilon_{11} dx_1 + f(x_2)$$

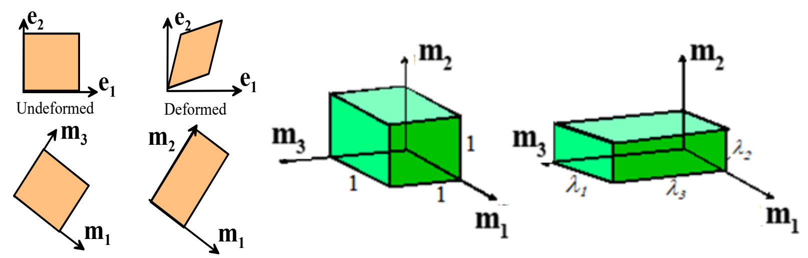
$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \Rightarrow u_2 = \int \varepsilon_{22} dx_2 + g(x_1)$$

Find *f,g* using $2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$

**Principal Strains and stretches**

In principal basis $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}\}$ strains are diagonal

$$\varepsilon_{ij}^{(\mathbf{m})} \equiv \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \qquad E_{ij}^{(\mathbf{m})} \equiv \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix}$$




Infinitesimal strain $\boldsymbol{\varepsilon}\mathbf{m}^{(i)} = e_i \mathbf{m}^{(i)}$

Lagrange strain $\mathbf{E}\mathbf{m}^{(i)} = \frac{1}{2}(\lambda_i^2 - 1)\mathbf{m}^{(i)}$

(eigenvalues – use usual method to find them)

# Concept Checklist

**5. Forces**

- Understand the concepts of external surface traction and internal body force;
- Understand the concept of internal traction inside a solid.
- Understand how Newton's laws imply the existence of the Cauchy stress tensor
- Be able to calculate tractions acting on an internal plane with given orientation from the Cauchy stress tensor
- Be able to integrate tractions exerted by stresses over a surface to find the resultant force
- Know the definition of principal stresses, be able to calculate values of principal stress and their directions, understand the physical significance of principal stresses
- Know the definition of Hydrostatic stress and von Mises stress
- Understand the use of stresses in simple failure criteria (yield and fracture)
- Understand the boundary conditions for stresses at an exterior surface

**6. Equations of motion for solids**

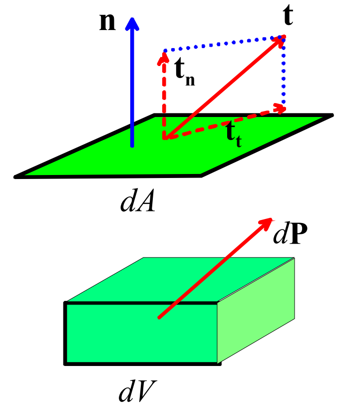
- Know the equations for linear momentum balance and angular momentum balance for a deformable solid
- Understand the significance of the small deformation approximation of the general equations of motion
- Know the equations of motion and static equilibrium for stress in Cartesian and polar coordinates
- Be able to check whether a stress field satisfies static equilibrium

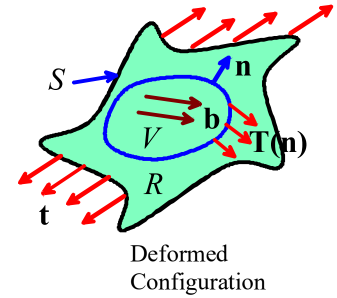
# Describing external and internal forces

## External Loading

Surface Traction $\mathbf{t} = \mathbf{t}_t + t_n \mathbf{n} = \lim_{dA \to 0} \frac{d\mathbf{P}}{dA}$

Body force (per unit mass) $\mathbf{b} = \lim_{dV \to 0} \frac{d\mathbf{P}}{\rho dV}$



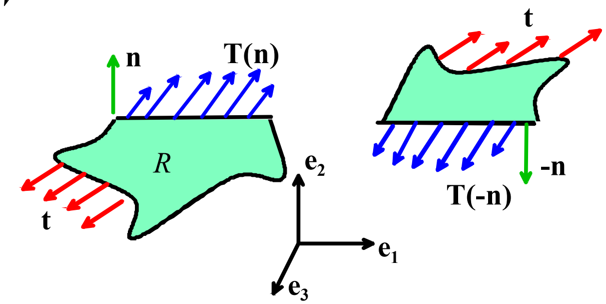

## Internal Traction Vector T(n)

Quantifies force per unit area at a point on internal plane

Traction depends on direction of normal to surface

Satisfies: $\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n})$

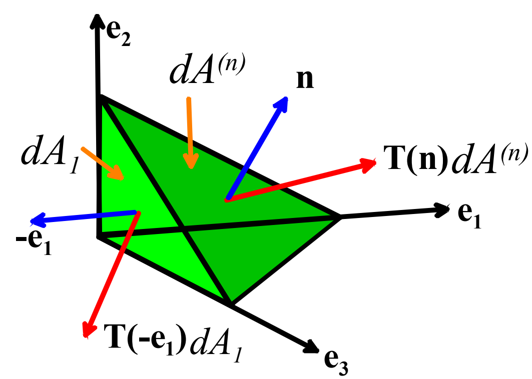
$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{e}_1)n_1 + \mathbf{T}(\mathbf{e}_2)n_2 + \mathbf{T}(\mathbf{e}_3)n_3$$




## Cauchy ("True") Stress Tensor

Definition (components): $\sigma_{ij} = T_j(\mathbf{e}_i)$

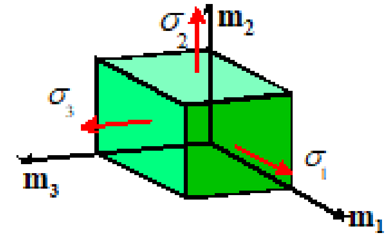
Then: $T_j(\mathbf{n}) = n_i \sigma_{ij}$ $\mathbf{T} = \mathbf{n}\boldsymbol{\sigma}$




Warning: Some texts use transpose of this definition $\mathbf{T} = \boldsymbol{\sigma}\mathbf{n}$

Cauchy stress (force per unit deformed area) is symmetric $\sigma_{ij} = \sigma_{ji}$ , so both are the same, but some other stresses eg nominal stress (force per unit undeformed area) are not, so be careful.

# Stresses




Principal stresses (eigenvalues of stress tensor)

$$\mathbf{n}^{(i)}\boldsymbol{\sigma} = \sigma_i \mathbf{n}^{(i)} \quad \text{or} \quad n_j^{(i)}\sigma_{jk} = \sigma_i n_k^{(i)} \quad (\text{no sum on } i)$$

If $\sigma_1 > \sigma_2 > \sigma_3$ then $\sigma_1$ is the largest stress acting normal to any plane

Hydrostatic stress $\sigma_h = \text{trace}(\boldsymbol{\sigma})/3 \equiv \sigma_{kk}/3$ $\qquad \sigma_h = (\sigma_1 + \sigma_2 + \sigma_3)/3$

Deviatoric stress $\sigma'_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$

Von Mises stress $\sigma_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$

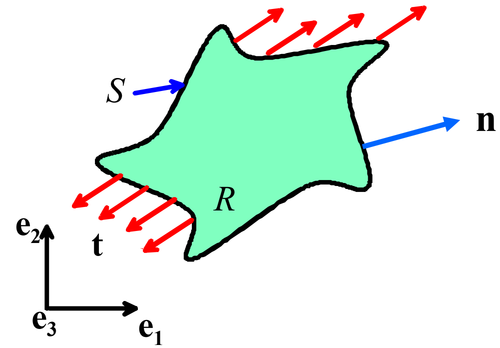
$$\sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right\}}$$

Failure criterion for brittle materials (approximate) $\sigma_1 < \sigma_{frac}$

Yield criterion for metals (Von Mises) $\sigma_e < Y$

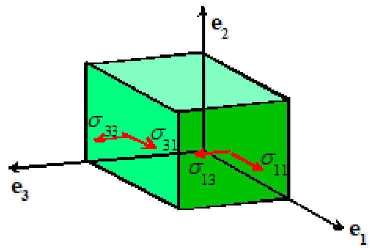
# Stresses

**Stresses near a boundary** $n_i \sigma_{ij} = t_j$ $\mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$




eg for $\mathbf{n} = \mathbf{e}_2$ $\mathbf{t} = \mathbf{0}$

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0$$



# Equations of motion

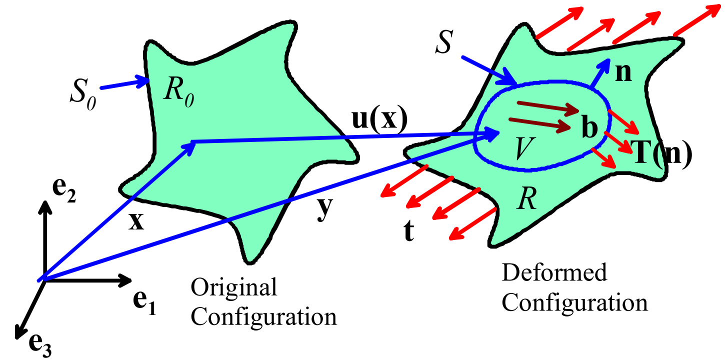
**Linear Momentum**

$$\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho a_j$$

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho \frac{dv_1}{dt}$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho \frac{dv_2}{dt}$$

$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho \frac{dv_3}{dt}$$

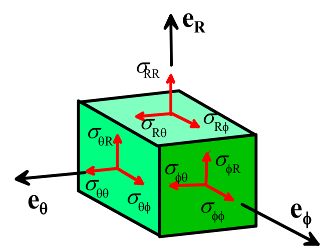


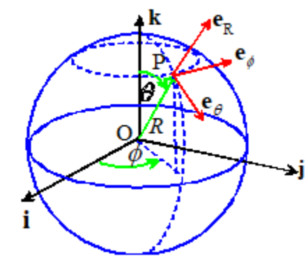

**Angular Momentum** $\sigma_{ij} = \sigma_{ji}$

**Small deformations:** replace **y** by **x** (approximate, but much easier to solve)

Spherical-polar coordinates

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{RR} & \sigma_{R\theta} & \sigma_{R\phi} \\ \sigma_{\theta R} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi R} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$



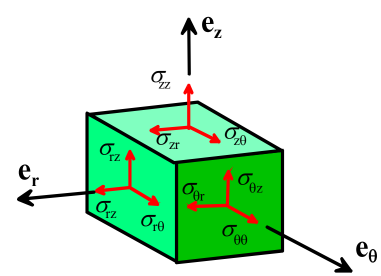



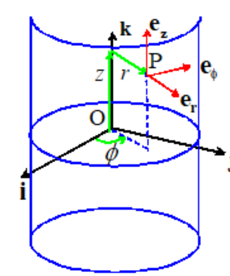

$$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \equiv \begin{bmatrix} \frac{\partial \sigma_{RR}}{\partial R} + 2\frac{\sigma_{RR}}{R} + \frac{1}{R}\frac{\partial \sigma_{\theta R}}{\partial \theta} + \cot\theta\frac{\sigma_{\theta R}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi R}}{\partial \phi} - \frac{1}{R}(\sigma_{\theta\theta} + \sigma_{\phi\phi}) \\ \frac{\partial \sigma_{R\theta}}{\partial R} + 2\frac{\sigma_{R\theta}}{R} + \frac{1}{R}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \cot\theta\frac{\sigma_{\theta\theta}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta R}}{R} - \cot\theta\frac{\sigma_{\phi\phi}}{R} \\ \frac{\partial \sigma_{R\phi}}{\partial R} + 2\frac{\sigma_{R\phi}}{R} + \frac{\sin\theta}{R}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos\theta\frac{\sigma_{\theta\phi}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{R}(\sigma_{\phi R} + \sigma_{\phi\theta}) \end{bmatrix} + \begin{bmatrix} \rho b_R \\ \rho b_\theta \\ \rho b_\phi \end{bmatrix} = \begin{bmatrix} \rho\frac{dv_R}{dt} \\ \rho\frac{dv_\theta}{dt} \\ \rho\frac{dv_\phi}{dt} \end{bmatrix}$$

Cylindrical-polar coordinates

$$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \equiv \begin{bmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zR}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta r}}{r} + \frac{\partial \sigma_{z\theta}}{\partial z} \\ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} \end{bmatrix} + \begin{bmatrix} \rho b_r \\ \rho b_\theta \\ \rho bz \end{bmatrix} = \begin{bmatrix} \rho\frac{dv_r}{dt} \\ \rho\frac{dv_\theta}{dt} \\ \rho\frac{dv_z}{dt} \end{bmatrix}$$

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$



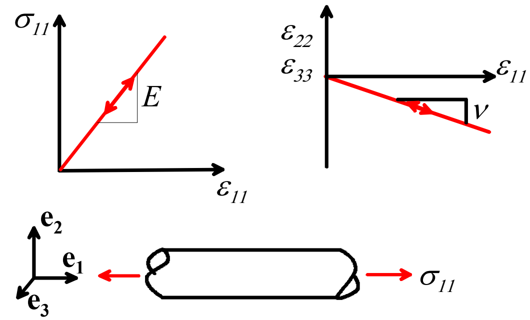



# Concept Checklist

**7. Stress-strain relations for elastic materials subjected to small strains**

- Understand the concept of an isotropic material
- Understand the assumptions associated with idealizing a material as linear elastic
- Know the stress-strain-temperature equations for an isotropic, linear elastic solid
- Understand how to simplify stress-strain temperature equations for plane stress or plane strain deformation
- Be familiar with definitions of elastic constants (Young's, shear, bulk and Lame moduli, Poissons ratio)
- Be able to calculate strain energy density of a stress or strain field in an elastic solid
- Be able to calculate stress/strain in an elastic solid subjected to uniform loading or temperature.

# Stress-strain-temperature relations for elastic solids

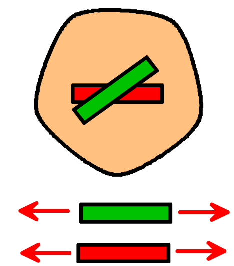



**Assumptions**

**Displacements/rotations are small** – we can use infinitesimal strain as our deformation measure

**Isotropy:** Material response is independent of orientation of specimen with respect to underlying material

**Elasticity:** Material behavior is perfectly reversible, and relation between stress, strain and temperature is linear



Stress-strain response equal

Then

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$

More generally

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha\Delta T\delta_{kl}\right) \qquad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}$$

## Strain Energy Density

Separate strain into elastic and thermal parts

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T$$

$$\varepsilon_{ij}^e = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{ij}^T = \alpha\Delta T\delta_{ij}$$

Strain energy density

$$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^e$$

$$U = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{kk}\sigma_{jj}$$

$$U = \frac{E}{2(1+\nu)}\varepsilon_{ij}^e\varepsilon_{ij}^e + \frac{E\nu}{2(1+\nu)(1-2\nu)}\varepsilon_{jj}^e\varepsilon_{kk}^e$$

# Useful elasticity formulas for isotropic materials

## Matrix form for stress-strain law (3D)

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

## For plane strain $\varepsilon_{33}=\varepsilon_{23}=\varepsilon_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{(1+\nu)}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}+(1+\nu)\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\sigma_{33}=\frac{E\nu(\varepsilon_{11}+\varepsilon_{22})}{(1-2\nu)(1+\nu)}+\frac{E\alpha\Delta T}{1-2\nu},\ \sigma_{13}=\sigma_{23}=0$$

## For plane stress $\sigma_{33}=\sigma_{23}=\sigma_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{(1-\nu)}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\varepsilon_{33}=-\frac{\nu}{E}(\sigma_{11}+\sigma_{22})+\alpha\Delta T$$

**Strain Energy Density**

$$U=\frac{1}{2}\left[\sigma_{11}\varepsilon_{11}^{e}+\sigma_{22}\varepsilon_{22}^{e}+\sigma_{33}\varepsilon_{33}^{e}+2\sigma_{12}\varepsilon_{12}+2\sigma_{13}\varepsilon_{13}+2\sigma_{23}\varepsilon_{23}\right]$$

$$\varepsilon_{11}^{e}=\varepsilon_{11}-\alpha\Delta T\quad \varepsilon_{22}^{e}=\varepsilon_{22}-\alpha\Delta T\quad \varepsilon_{33}^{e}=\varepsilon_{33}-\alpha\Delta T$$

# Relations between elastic constants

| | LAME MODULUS $\lambda$ | SHEAR MODULUS $\mu$ | YOUNG'S MODULUS $E$ | POISSON'S RATIO $\nu$ | BULK MODULUS $K$ |
|---|---|---|---|---|---|
| $\lambda, \mu$ | | | $\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ | $\frac{\lambda}{2(\lambda+\mu)}$ | $\frac{3\lambda+2\mu}{3}$ |
| $\lambda, E$ | | Irrational | | Irrational | Irrational |
| $\lambda, \nu$ | | $\frac{\lambda(1-2\nu)}{2\nu}$ | $\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$ | | $\frac{\lambda(1+\nu)}{3\nu}$ |
| $\lambda, K$ | | $\frac{3(K-\lambda)}{2}$ | $\frac{9K(K-\lambda)}{3K-\lambda}$ | $\frac{\lambda}{3K-\lambda}$ | |
| $\mu, E$ | $\frac{\mu(2\mu-E)}{E-3\mu}$ | | | $\frac{E-2\mu}{2\mu}$ | $\frac{\mu E}{3(3\mu-E)}$ |
| $\mu, \nu$ | $\frac{2\mu\nu}{1-2\nu}$ | | $2\mu(1+\nu)$ | | $\frac{2\mu(1+\nu)}{3(1-2\nu)}$ |
| $\mu, K$ | $\frac{3K-2\mu}{3}$ | | $\frac{9K\mu}{3K+\mu}$ | $\frac{3K-2\mu}{2(3K+\mu)}$ | |
| $E, \nu$ | $\frac{\nu E}{(1+\nu)(1-2\nu)}$ | $\frac{E}{2(1+\nu)}$ | | | $\frac{E}{3(1-2\nu)}$ |
| $E, K$ | $\frac{3K(3K-E)}{9K-E}$ | $\frac{3EK}{9K-E}$ | | $\frac{3K-E}{6K}$ | |
| $\nu, K$ | $\frac{3K\nu}{(1+\nu)}$ | $\frac{3K(1-2\nu)}{2(1+\nu)}$ | $3K(1-2\nu)$ | | |

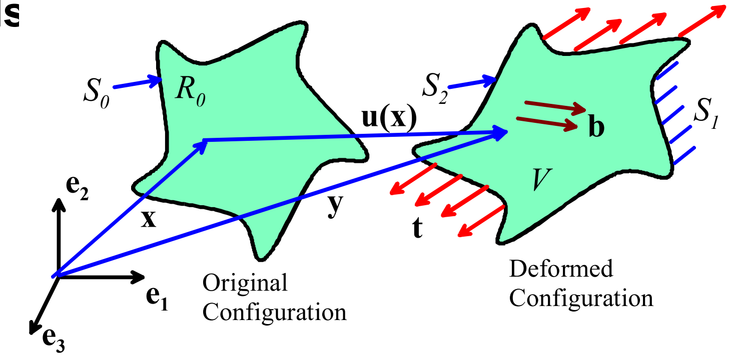
# Concept Checklist

**8. Analytical solutions to static problems for linear elastic solids**

- Know the general equations (strain-displacement/compatibility, stress-strain relations, equilibrium) and boundary conditions that are used to calculate solutions for elastic solids
- Understand general features of solutions to elasticity problems: (1) solutions are linear; (2) solutions can be superposed; (3) Saint-Venants principle
- Know how to simplify the equations for spherically symmetric solids (using polar coords)
- Be able to calculate stress/strain in spherically or cylindrically symmetric solids under spherical/cylindrical symmetric loading by hand
- Understand how the Airy function satisfies the equations of equilibrium and compatibility for an elastic solid
- Be able to check whether an Airy function is valid, and be able to calculate stress/strain/displacements from an Airy function and check that the solution satisfies boundary conditions

# Solutions for elastic solids

**Static boundary value problems for linear elastic solids**




**Assumptions:**

1. Small displacements
2. Isotropic, linear elastic material

**Given:**

1. Traction or displacement on all exterior surfaces
2. Body force and temperature distribution

**Find:** $[u_i, \varepsilon_{ij}, \sigma_{ij}]$

**Governing Equations:**

1. Strain-displacement relation (you can use the compatibility equation instead)

$$\varepsilon_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2 \qquad \boldsymbol{\varepsilon} = \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right]/2$$

2. Stress-strain law

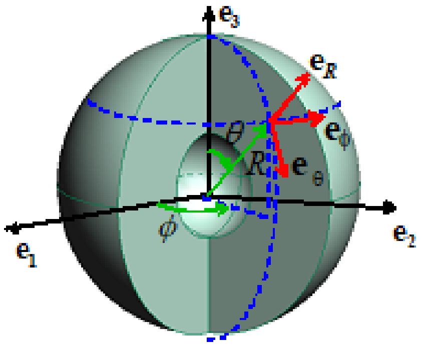
$$\sigma_{ij} = \frac{E}{1+\nu}\left( \varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \right) - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij} \qquad \boldsymbol{\sigma} = \frac{E}{1+\nu}\left( \boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu} trace(\boldsymbol{\varepsilon})\mathbf{I} \right) - \frac{E\alpha\Delta T}{(1-2\nu)}\mathbf{I}$$

3. Equilibrium $\dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = 0 \qquad \nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0}$

4. Boundary conditions on external surfaces
   1. Where displacements are prescribed $u_i = u_i^* \qquad \mathbf{u} = \mathbf{u}^*$
   2. Where tractions are prescribed $n_j \sigma_{ji} = t_i \qquad \mathbf{n}\boldsymbol{\sigma} = \mathbf{t}$

# Solutions for elastic solids

## Spherically symmetric solids




Position, displacement, body force

$$\mathbf{x} = R\mathbf{e}_R$$
$$\mathbf{u} = u(R)\mathbf{e}_R$$
$$\mathbf{b} = \rho_0 b(R)\mathbf{e}_R$$

Stress/strain

$$\sigma \equiv \begin{bmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \varepsilon \equiv \begin{bmatrix} \varepsilon_{RR} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{\phi\phi} \end{bmatrix}$$

$$\varepsilon_{RR} = \frac{du}{dR} \qquad \varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u}{R}$$

$$\sigma_{RR} = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\varepsilon_{RR} + \nu\varepsilon_{\theta\theta} + \nu\varepsilon_{\phi\phi}\right\} - \frac{E\alpha\Delta T}{1-2\nu}$$
$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{E}{(1+\nu)(1-2\nu)}\left\{\varepsilon_{\theta\theta} + \nu\varepsilon_{RR}\right\} - \frac{E\alpha\Delta T}{1-2\nu}$$

Equilibrium

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho_0 b_R = 0$$

$$\frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2u}{R^2} = \frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2 u\right)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{d\Delta T}{dR} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0 b(R)$$

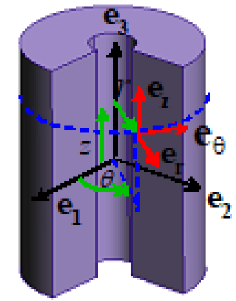
Boundary conditions

$$u_R(a) = g_a \qquad u_R(b) = g_b$$

or $\sigma_{RR}(a) = t_a \qquad \sigma_{RR}(b) = t_b$

# Solutions for elastic solids

## Cylindrically symmetric solids



Plane strain, or generalized plane strain

Position, displacement, body force

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$$
$$\mathbf{u} = u(r)\mathbf{e}_r + \varepsilon_{zz} z\mathbf{e}_z$$
$$\mathbf{b} = \rho_0 b(r)\mathbf{e}_r$$

Stress/strain

$$\sigma \equiv \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \qquad \varepsilon \equiv \begin{bmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \qquad \varepsilon_{rr} = \frac{du}{dr} \qquad \varepsilon_{\theta\theta} = \frac{u}{r}$$

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Plane strain

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} - \frac{E\alpha\Delta T}{1-\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Plane stress

Equilibrium

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho_0 b_r = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{\partial \Delta T}{\partial r} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0(b+\omega^2 r)$$

Plane strain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru)\right\} = \alpha(1+\nu)\frac{\partial \Delta T}{\partial r} - \frac{(1-\nu^2)}{E}\rho_0(b+\omega^2 r)$$

Plane stress

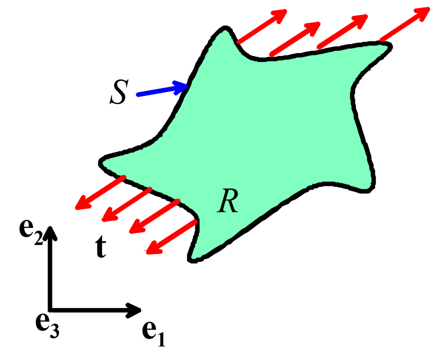
Boundary conditions

$$u_r(a) = g_a \qquad u_r(b) = g_b$$

or $\sigma_{rr}(a) = t_a \qquad \sigma_{rr}(b) = t_b$

# Airy Function solution to elasticity problems

Airy Function $\nabla^4\phi \equiv \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\phi}{\partial x_1^4} = 0$



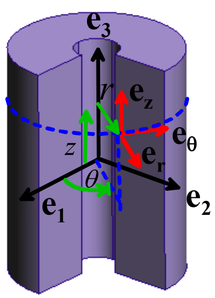

Stress $\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2\phi}{\partial x_1\partial x_2}$

$\sigma_{33} = 0$ (Plane Stress)

$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$ (Plane Strain)

$\sigma_{23} = \sigma_{13} = 0$

Airy Function $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \phi = 0$




Stress $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$

Strain $\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix} = \frac{(1+\nu)}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$ Plane Strain

$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$ Plane Stress

Displacement $\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$

# Proof of the Airy Representation

**Airy Function**

$$\nabla^4\phi \equiv \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial x_1^4} = 0$$

**Stress**

$$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = 0 \qquad \text{(Plane Stress)}$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \qquad \text{(Plane Strain)}$$

$$\sigma_{23} = \sigma_{13} = 0$$

**Strain**

$$\varepsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(1+\beta\nu)(\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(1+\beta\nu)(\sigma_{11} + \sigma_{22})$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \qquad \beta = 1 \ \text{(Plane Strain)} \qquad \beta = 0 \ \text{(Plane Stress)}$$

**Equilibrium**

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \qquad \frac{\partial}{\partial x_1}\left(\frac{\partial^2\phi}{\partial x_2^2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial^2\phi}{\partial x_1 \partial x_2}\right) = 0 \quad ✓$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \qquad \frac{\partial}{\partial x_1}\left(-\frac{\partial^2\phi}{\partial x_1 \partial x^2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial^2\phi}{\partial x_1^2}\right) = 0 \quad ✓$$

**Compatibility**

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \qquad \frac{1+\nu}{E}\left(\frac{\partial^2 \sigma_{11}}{\partial x_2^2} + \frac{\partial^2 \sigma_{22}}{\partial x_1^2}\right) - \frac{\nu}{E}(1+\beta\nu)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11} + \sigma_{22}) - 2\frac{1+\nu}{E}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^4\phi}{\partial x_2^4} + \frac{\partial^4\phi}{\partial x_1^4} - \frac{\nu(1+\beta\nu)}{1+\nu}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2}\right) + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} = 0$$

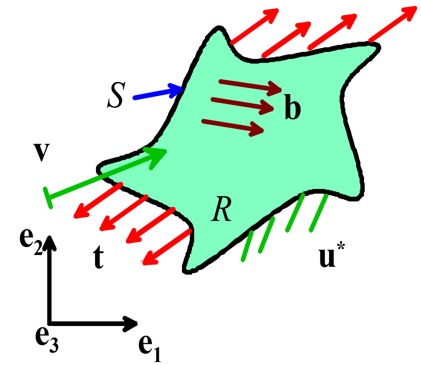
$$\frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4\phi}{\partial x_1^4} = 0 \quad ✓$$

# Concept Checklist

**9. Energy methods for linear elastic solids**

- Understand the definition of a kinematically admissible displacement field
- Know the formulas for potential energy of 3D solids, strings, beams, membranes and plates
- Know the principle of minimum potential energy for elastic solids
- Be able to use energy methods to estimate the stiffness of a solid
- Be able to use the Rayleigh-Ritz method to find approximate solutions to elastic boundary value problems

# Principle of Minimum Potential Energy




**Assumptions**:

1. Elastic material
2. Small displacements
3. Static equilibrium
4. Boundary conditions $\mathbf{u} = \mathbf{u}^*$ on $S_1$ $\mathbf{n}\boldsymbol{\sigma} = \mathbf{t}$ on $S_2$

**Definitions**:

1. Kinematically admissible displacement field: any differentiable displacement vector satisfying $\mathbf{v} = \mathbf{u}^*$ wherever displacements are known
2. Actual displacement field (the one that satisfies equilibrium within the solid and traction boundary conditions on surfaces) $\mathbf{u}$
3. Strain energy density $U$
4. Potential energy

$$\Pi(\mathbf{v}) = \int_V \mathrm{U}(\mathbf{v})dV - \int_V \mathbf{b}\cdot\mathbf{v}dV - \int_{S2} \mathbf{t}\cdot\mathbf{v}dA$$

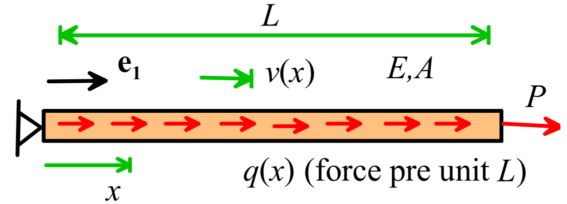
**Principle of minimum potential energy** $\Pi(\mathbf{v}) \geq \Pi(\mathbf{u})$

Among all guesses for the displacement field, the best guess is the one with the smallest $\Pi$

# Useful formulas for potential energy

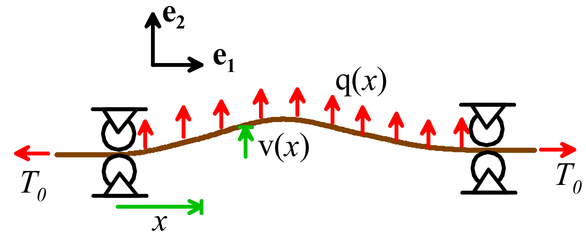
1-D axially loaded bar

$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{dv}{dx}\right)^2 dx - \int_0^L q(x)v(x)dx - Pv(L)$$



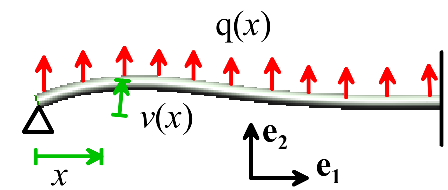

1-D Tensioned cable

$$\Pi = T_0 \int_0^L \frac{1}{2} \left(\frac{dv}{dx}\right)^2 dx - \int_0^L q(x)v(x)dx$$



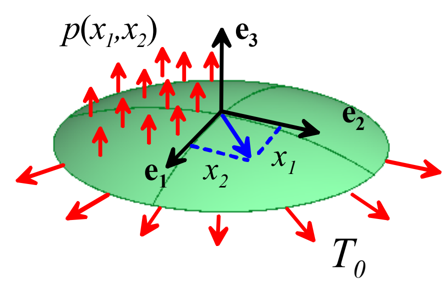

1-D Euler-Bernoulli beam

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L q(x)v(x)dx$$



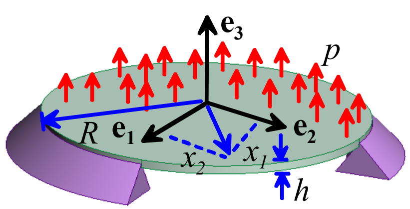

2-D biaxially stretched membrane

$$\Pi = \int_A \frac{1}{2} T_0 \left[ \left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 \right] dA - \int_A q(x_1,x_2)v(x_1,x_2)dA$$




2-D Kirchhoff plate

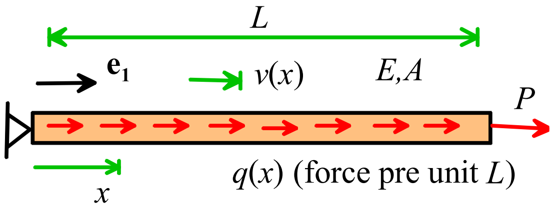
$$\Pi = \frac{Eh^3}{12(1-\nu)} \int_A \frac{1}{2} \left[ \left( \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} \right)^2 - 2(1-\nu) \left( \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \left( \frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right) \right] dA - \int_A q(x_1,x_2)v(x_1,x_2)dA$$



# Rayleigh-Ritz Approximation

**Example:** 1-D axially loaded bar

$$\Pi = \int_0^L \frac{1}{2} EA \left( \frac{dv}{dx} \right)^2 dx - \int_0^L q(x) v(x) dx - P v(L)$$




**Approximation:** $v(x) = \sum_{i=1}^{N} a_i f_i(x)$   $f_i(x)$ - Basis functions (any complete set of interpolation functions)

$f_i(x) = x^{i-1}$ is an example

**1. Satisfy Boundary Conditions:** $v(0) = \sum_{i=1}^{N} a_i f_i(0) = 0$

**2. Eliminate some subset of** $a_i$

**3. Calculate PE** $\Pi = \int_0^L \frac{1}{2} EA \left( \sum_{i=2}^{N} a_i \frac{df_i}{dx} \right)^2 dx - \int_0^L \sum_{i=2}^{N} a_i f_i(x) q(x) dx - P \sum_{i=2}^{N} a_i f_i(L)$

**4. Minimize** $\frac{\partial \Pi}{\partial a_i} = 0$

**4. Solve for remaining** $a_i$

# Concept Checklist

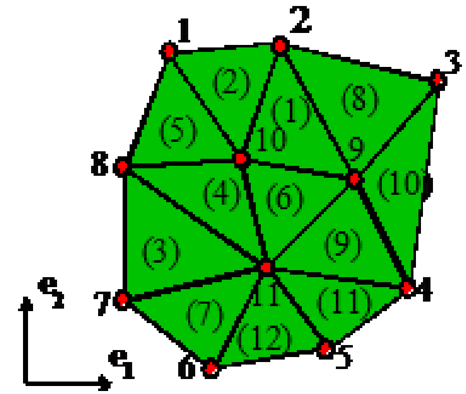
**10. Implementing the finite element method for linear elastostatics**

- Understand the connection between elastostatic FEA and the Rayleigh-Ritz method
- Understand how the finite element mesh is used to interpolate a displacement field
- Be able to calculate the strain energy of an element
- Be able to calculate the element stiffness matrix
- Understand how to sum the element stiffness into a global stiffness matrix
- Be able to calculate the potential energy of external forces acting on a mesh
- Be able to re-write the potential energy of external loads in terms of the global force vector
- Understand how to minimize the potential energy to derive a linear system of equations for unknown displacements
- Know how to modify the system of linear equations so as to enforce constraints on displacements
- Know how to post-process the solution to calculate stresses and strains
- Be familiar with the structure of a linear elastic finite element code

- Understand the effects of improper boundary conditions on the FE equation system
- Be aware of ‘locking’ of FE equations with certain element formulations for incompressible materials

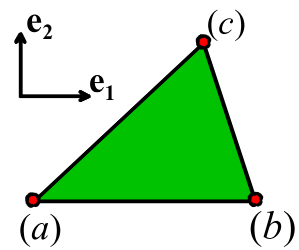
# Simple FEA for plane linear elasticity

- **Approach**: compute displacement field in an elastic solid by
  - Interpolating displacement field
  - Calculating total potential energy of solids in terms of discrete displacements
  - Minimize potential energy




- **Interpolation** – constant strain triangles

$$u_i(x_1,x_2) = u_i^{(a)} N^a(x_1,x_2) + u_i^{(b)} N^b(x_1,x_2) + u_i^{(c)} N^c(x_1,x_2)$$




$$N^a(x_1,x_2) = \frac{\left(x_2 - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1 - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}{\left(x_2^{(a)} - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1^{(a)} - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}$$

$$N^b(x_1,x_2) = \frac{\left(x_2 - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1 - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}{\left(x_2^{(b)} - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1^{(b)} - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}$$

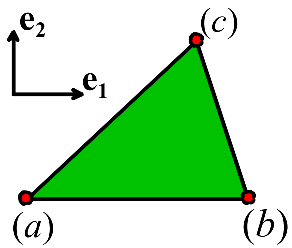
$$N^c(x_1,x_2) = \frac{\left(x_2 - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1 - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}{\left(x_2^{(c)} - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1^{(c)} - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}$$

- **Potential Energy** $\Pi = \int_A U dA - \int_{S_2} \mathbf{t}^* \cdot \mathbf{u} ds$

# Calculating strain energy density in an element

**Strains:**

$$\underline{\varepsilon} = [B]\underline{u}^{\text{element}} \equiv \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 & \frac{\partial N_b}{\partial x_1} & 0 & \frac{\partial N_c}{\partial x_1} & 0 \\ 0 & \frac{\partial N_a}{\partial x_2} & 0 & \frac{\partial N_b}{\partial x_2} & 0 & \frac{\partial N_c}{\partial x_2} \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} & \frac{\partial N_b}{\partial x_2} & \frac{\partial N_b}{\partial x_1} & \frac{\partial N_c}{\partial x_2} & \frac{\partial N_c}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1^{(a)} \\ u_2^{(a)} \\ u_1^{(b)} \\ u_2^{(b)} \\ u_1^{(c)} \\ u_2^{(c)} \end{bmatrix}$$




(Strains are constant for linear triangles; not true for general elements)

**Stresses (plane strain – can use equivalent plane stress relations as well):**

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

**Strain energy density** $U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\begin{bmatrix} \varepsilon_{11} & \varepsilon_{22} & 2\varepsilon_{12} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$ (Constant in linear triangles)

**Total strain energy** $W^{\text{element}} = \frac{1}{2}\underline{u}^{\text{element}T}\left(A_{element}[B]^T[D][B]\right)\underline{u}^{\text{element}}$

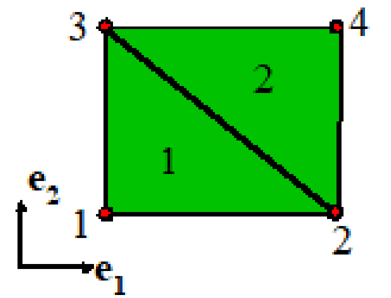
**Define element stiffness** $[K^{\text{element}}] = A_{element}[B]^T[D][B]$ Symmetric 6x6 matrix

$$W^{\text{element}} = \frac{1}{2}\underline{u}^{\text{element}T}[K^{\text{element}}]\underline{u}^{\text{element}}$$

# Calculating the total strain energy

**Sum strain energy over elements:**

$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}^T \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & \cdots & k_{16}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & & \\ \vdots & & \ddots & \\ k_{61}^{(1)} & & & k_{66}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}^T \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & \cdots & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & & \\ \vdots & & \ddots & \\ k_{61}^{(2)} & & & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$




**Rewrite each term in terms of global displacement vector:**

$$W = \frac{1}{2}\left[u_1^{(1)} \; u_2^{(1)} \; u_1^{(2)} \; u_2^{(2)} \; u_1^{(3)} \; u_2^{(3)} \; u_1^{(4)} \; u_2^{(4)}\right] \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & & \cdots & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & & 0 & 0 \\ & & k_{33}^{(1)} & k_{34}^{(1)} & & & 0 & 0 \\ & & k_{43}^{(1)} & k_{44}^{(1)} & & & 0 & 0 \\ & & k_{53}^{(1)} & & & & 0 & 0 \\ & & \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} + \frac{1}{2}\left[u_1^{(1)} \; u_2^{(1)} \; u_1^{(2)} \; u_2^{(2)} \; u_1^{(3)} \; u_2^{(3)} \; u_1^{(4)} \; u_2^{(4)}\right] \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & k_{11}^{(2)} & k_{12}^{(2)} & & & & \\ 0 & 0 & k_{21}^{(2)} & k_{22}^{(2)} & & & & \\ 0 & 0 & k_{31}^{(2)} & & & & & \\ 0 & 0 & & & & \ddots & & \\ 0 & 0 & \vdots & & & & & k_{56}^{(2)} \\ 0 & 0 & & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

**Combine to create global stiffness:**

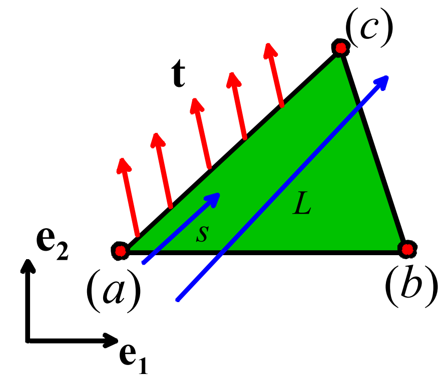
$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}^T \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & \cdots & \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & \\ & & k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & & \\ & & k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & & \\ & & k_{53}^{(1)} + k_{31}^{(2)} & & \ddots & \\ & & \vdots & & & k_{56}^{(2)} \\ & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

$$W = \frac{1}{2}\underline{u}^{gT}\left[K\right]\underline{u}^{g}$$

# Calculating the potential energy of external forces

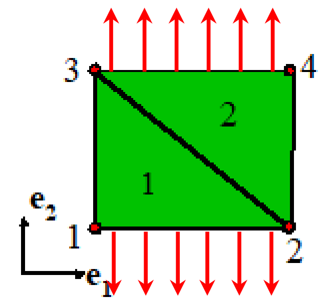
**Total PE:**

$$\Pi = \int_A U dA - \int_{S_2} \mathbf{t} \cdot \mathbf{u} ds$$




**Sum over all loaded element faces:**

$$\rho^{el} = \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 \end{bmatrix} \cdot \begin{bmatrix} r_1^1 & r_2^1 & r_1^1 & r_2^1 \end{bmatrix} + \begin{bmatrix} u_1^3 & u_2^3 & u_1^4 & u_2^4 \end{bmatrix} \cdot \begin{bmatrix} r_1^2 & r_2^2 & r_1^2 & r_2^2 \end{bmatrix}$$




**Rewrite in terms of global displacement:**

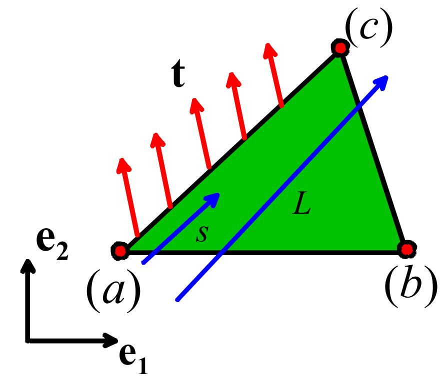
$$\rho^{TOT} = \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 & u_1^3 & u_2^3 & u_1^4 & u_2^4 \end{bmatrix} \cdot \begin{bmatrix} r_1^1 & r_2^1 & r_1^1 & r_2^1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 & u_1^3 & u_2^3 & u_1^4 & u_2^4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & r_1^2 & r_2^2 & r_1^2 & r_2^2 \end{bmatrix}$$

**Combine:**

$$\rho^{TOT} = \begin{bmatrix} u_1^1 & u_2^1 & u_1^2 & u_2^2 & u_1^3 & u_2^3 & u_1^4 & u_2^4 \end{bmatrix} \cdot \begin{bmatrix} r_1^1 & r_2^1 & r_1^1 & r_2^1 & r_1^2 & r_2^2 & r_1^2 & r_2^2 \end{bmatrix}$$
$$= \underline{u}^{gT} \cdot \underline{r}$$

# Minimizing PE and constraining prescribed displacements

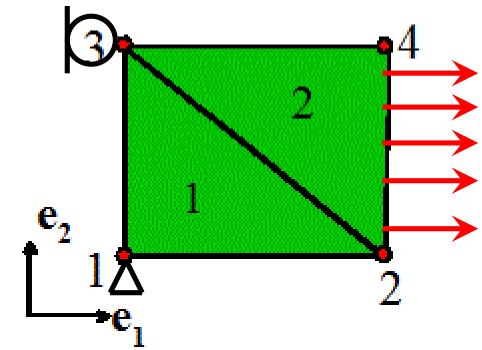
**Potential energy :** $$\Pi = \int_A U dA - \int_{S_2} \mathbf{t}\cdot\mathbf{u}\, ds \approx \frac{1}{2}\underline{u}^{gT}[K]\underline{u}^{g} - \underline{u}^{gT}\cdot\underline{r}$$




**Minimize:** $$\frac{\partial \Pi}{\partial \underline{u}^{g}} = [K]\underline{u}^{g} - \underline{r} \;\; = \underline{0}$$

Linear equations – but without displacement BCs the stiffness matrix K will be singular so the equations may not have a solution, and if they do the solution will not be unique.

**Prescribing displacements:**




Modify rows corresponding to constrained nodes to insert equations constraining displacements

Example: enforce $u_2 = \Delta$ for node number 1

Original equations

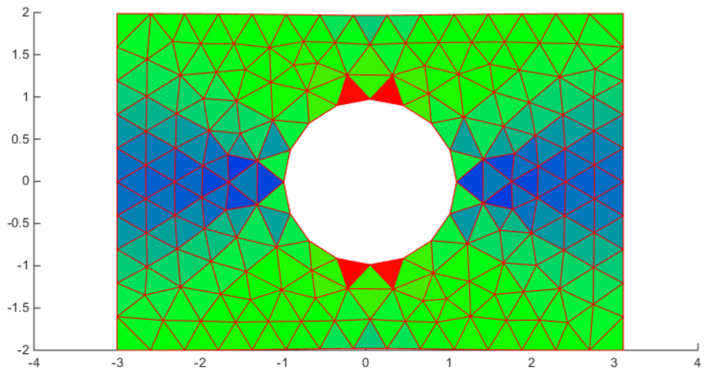
$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{12N} \\ k_{21} & k_{22} & & k_{22N} \\ \vdots & & \ddots & \\ k_{2N1} & k_{2N2} & & k_{2N2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_4 \end{bmatrix}$$

Modified equations

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1\,2N} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ k_{2N\,1} & k_{2N\,2} & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ \Delta \\ \vdots \\ r_4 \end{bmatrix}$$

# Structure of a basic FEA code

- Data defining problem (GUI or input file):
  - Material properties
  - Nodal coordinates
  - Element connectivity
  - List of nodes with prescribed DOF
  - List of elements with loaded faces




- Loop over elements
  - Compute element stiffness, add to global stiffness
- Loop over elements with loaded faces
  - Compute element force vector, add to global force vector
- Modify stiffness and RHS to impose prescribed disps.
- Solve FEA equations for unknown nodal displacements
- Post-processing – compute element strains & stresses

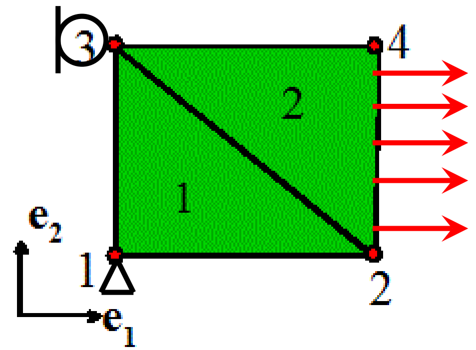
# Improper constraints lead to singular stiffness matrix

eigenvecs =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | -0.3152 | -0.4728 | 0 | -0.3152 | 0.4781 | 0.5908 |
| 0 | 0 | 0.6043 | -0.4559 | 0 | 0.6043 | 0.0433 | 0.2449 |
| 0 | 0 | 0.0866 | 0.6137 | 0 | 0.0866 | -0.2079 | 0.7518 |
| 0 | 0 | -0.7071 | -0.0000 | 0 | 0.7071 | -0.0000 | 0.0000 |
| 0 | 0 | -0.1673 | -0.4381 | 0 | -0.1673 | -0.8522 | 0.1606 |
| 0.4779 | 0 | 0.4907 | 0.5123 | 0 | 0.0000 | -0.0080 | 0.5179 |
| 0 | 0.4779 | 0.0000 | 0.5123 | 0 | 0.4907 | -0.0080 | 0.5179 |
| 0 | 0 | -0.0039 | -0.5103 | 0.4759 | -0.0039 | 0.4928 | -0.5198 |

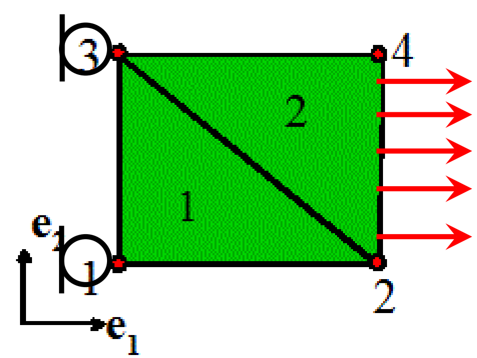
eigenvals =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 199.8263 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 125.4412 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 13.8635 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 38.4615 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 55.0998 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.0000 |




eigenvecs =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0.3791 | -0.3791 | -0.3791 | 0 | -0.4610 | 0.3791 | 0.4610 |
| 0 | -0.4523 | 0.3015 | -0.4523 | 0 | 0.4523 | 0.3015 | 0.4523 |
| 0 | -0.5000 | -0.5000 | 0.5000 | 0 | -0.0000 | 0.5000 | 0.0000 |
| 0 | -0.2132 | -0.6396 | -0.2132 | 0 | 0.2132 | -0.6396 | 0.2132 |
| 0 | -0.3260 | 0.3260 | 0.3260 | 0 | -0.5361 | -0.3260 | 0.5361 |
| 0 | 0.5000 | 0.0000 | 0.5000 | 0 | 0.5000 | 0.0000 | 0.5000 |
| 0.5638 | -0.2906 | 0.5789 | 0.2929 | 0 | -0.2984 | -0.0046 | 0.2961 |
| 0 | 0.2929 | -0.0046 | -0.2906 | 0.5638 | 0.2961 | 0.5789 | -0.2984 |




eigenvals =

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 216.4663 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 153.8462 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 76.9231 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 48.0769 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 23.9183 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | -0.0000 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1.0000 |

Zero eigenvalue
Stiffness is singular!

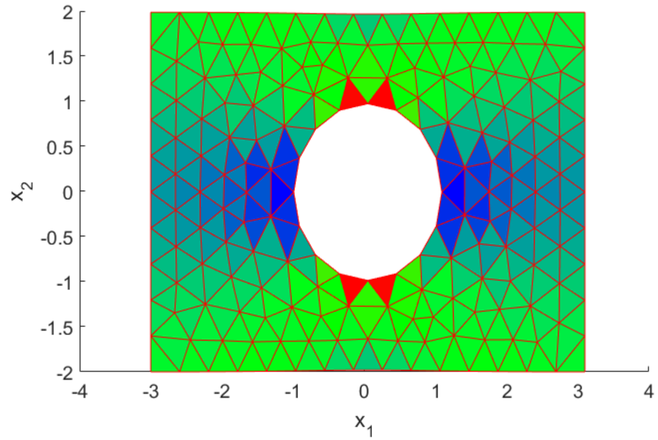
```
> In fem_conststrain_triangles (line 93)
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =
2.771117e-17.
```

# Volumetric locking in near-incompressible materials

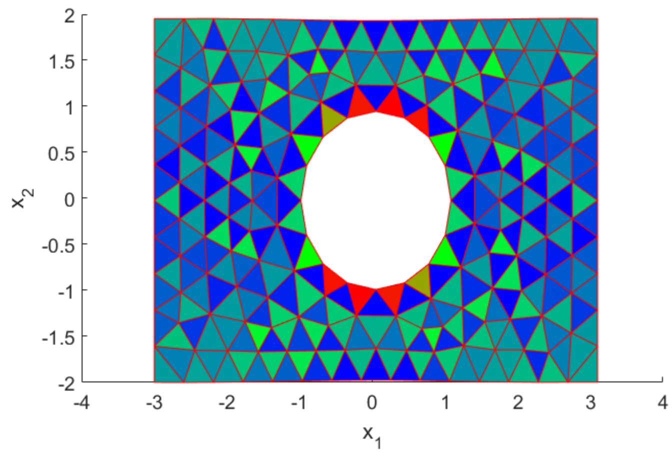
Example problem: plane strain strip with central hole

Contours show $\sigma_{11}$

**Results for** $\nu = 0.3$




**Results for** $\nu = 0.499$




Spurious pressure fluctuations – this happens because the elements become very stiff

Constant strain triangles always give incorrect results for near incompressible materials – there is no fix.

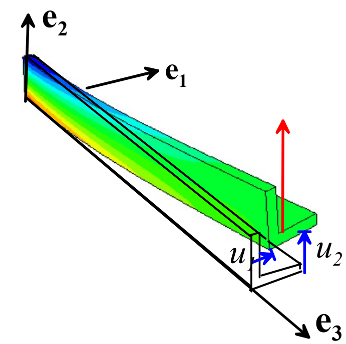
For other element types, reduced integration is used to correct volumetric locking. Hybrid elements are specially designed to be used for near incompressible materials

# Concept Checklist

**11. Approximate theories for solids with special shapes – beams**

- Understand how displacements and strains are approximated in straight beams
- Understand the difference between Euler-Bernoulli and Timoshenko beams
- Be able to calculate area moments of inertia of a beam cross-section
- Understand constraints and loading that can be imposed on beams
- Know the equations of linear and angular momentum for transverse motion of beams
- Know the differential equations for displacement fields in beams
- Know the simplified equation for strings and beams without axial force
- Be able to calculate deformed shapes of strings and beams in static equilibrium

# Beams




**Goal:** Calculate (1) Displacement of centroid $\mathbf{u}(x_3) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$

(2) Rotation of x-section $\boldsymbol{\theta}(x_3) = \theta_1\mathbf{e}_1 + \theta_2\mathbf{e}_2 + \theta_3\mathbf{e}_3$

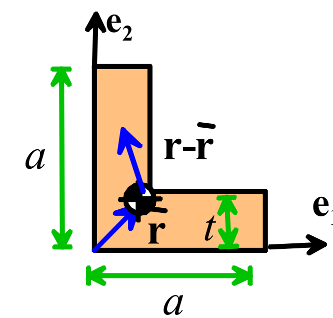
(3) Curvature vector $\boldsymbol{\kappa}(x_3) = d\boldsymbol{\theta} / dx_3$

Neglect twist $\theta_3$ here, but ABAQUS will include twist

**Section properties:**

$$A = \int_A dA \qquad \bar{\mathbf{r}} = \frac{1}{A}\int_A (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)dA$$

$$\mathbf{I} = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \qquad I_{11} = \int_A (x_2 - \bar{r}_2)^2 dA \qquad I_{22} = \int_A (x_1 - \bar{r}_1)^2 dA \qquad I_{12} = \int_A (x_1 - \bar{r}_1)(x_2 - \bar{r}_2)dA$$



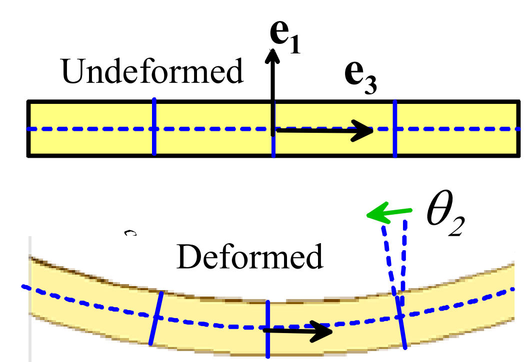

**Deformation:**

Euler-Bernoulli theory (no shear): $\theta_1 = -du_2 / dx_3 \qquad \theta_2 = du_1 / dx_3$

(Timoshenko theory allows x-sect to rotate relative to neutral axis)

Axial strain: $\varepsilon_{33} = \kappa_1(x_2 - \bar{r}_2) - \kappa_2(x_1 - \bar{r}_1)$

(Timoshenko beam has shear strains)



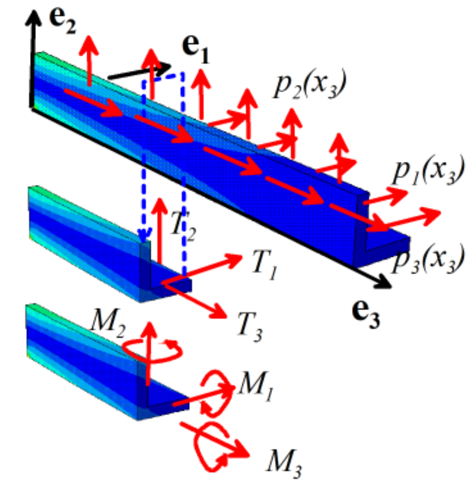

**Stresses:** $\sigma_{33} = E\varepsilon_{33}$

Other stresses zero in E-B beams

(Timoshenko beams have shear stresses)

# Beams

**Internal Forces (forces/moments on section normal to $\mathbf{e}_3$ :**




Force vector $\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3$

Moment vector $\mathbf{M} = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3$

$$M_1 = \int_A \sigma_{33}(x_2 - \overline{r}_2)dA \qquad M_2 = -\int_A \sigma_{33}(x_1 - \overline{r}_1)dA$$

No twist means $M_3 = 0$

**Moment-Curvature relations:**

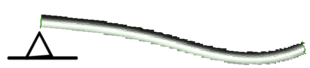
$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E\begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix}\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}$$

**Equations of motion:**

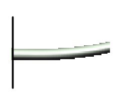
**F=ma:** $$\frac{dT_i}{dx_3} + p_i = \rho A a_i$$

Angular Momentum: $$\frac{dM_1}{dx_3} - T_2 - \theta_1 T_3 = 0 \qquad \frac{dM_2}{dx_3} + T_1 - \theta_2 T_3 = 0$$

**Boundary Conditions (at ends):**

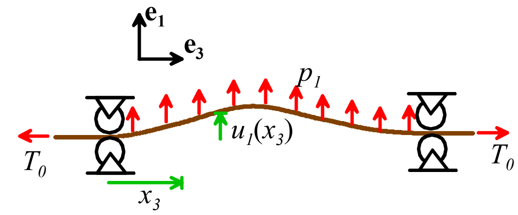
Fixed end: $\mathbf{u} = \mathbf{0}$ 

Free to move: $\mathbf{T} = \mathbf{0}$

Clamped end: $\boldsymbol{\theta} = \mathbf{0}$ 

Free to rotate: $\mathbf{M} = \mathbf{0}$

# Limiting Cases of Beam Equations




**Zero bending resistance** (string): $EI/(L^2T_3) << 1$

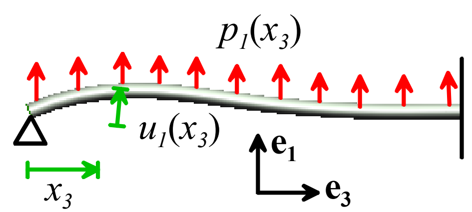
Axial Force $\frac{dT_3}{dx_3} + p_3 \approx \rho A \frac{d^2u_3}{dt^2}$

Transverse motion $\frac{d}{dx_3}\left(T_3\frac{du_1}{dx_3}\right) + p_1 = \rho A\frac{d^2u_1}{dt^2}$ $\quad \frac{d}{dx_3}\left(T_3\frac{du_2}{dx_3}\right) + p_2 = \rho A\frac{d^2u_2}{dt^2}$

Internal forces: $T_2 = T_0\frac{du_2}{dx_3}$ $\quad T_1 = T_0\frac{du_1}{dx_3}$

Boundary conditions: Either: $\mathbf{u} = \mathbf{u}^*$ $\quad x_3 = 0 \quad x_3 = L$ Or: $\mathbf{T} = -\mathbf{P}(0) \quad x_3 = 0 \quad \mathbf{T} = \mathbf{P}(L) \quad x_3 = L$

**Zero axial force** (simple beam): $EI/(L^2T_3) >> 1$




Transverse motion

$$E\left(I_{22}\frac{d^4u_1}{dx_3^4} + I_{12}\frac{d^4u_2}{dx_3^4}\right) + \rho A\frac{d^2u_1}{dt^2} = p_1$$

$$E\left(I_{12}\frac{d^4u_1}{dx_3^4} + I_{11}\frac{d^4u_2}{dx_3^4}\right) + \rho A\frac{d^2u_2}{dt^2} = p_2$$

Internal forces: $\mathbf{M} = E\mathbf{I}\boldsymbol{\kappa}$ $\quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E\begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix}\begin{bmatrix} -d^2u_2/dx_3^2 \\ d^2u_1/dx_3^2 \end{bmatrix}$ $\quad \frac{dM_1}{dx_3} - T_2 = 0 \quad \frac{dM_2}{dx_3} + T_1 = 0$

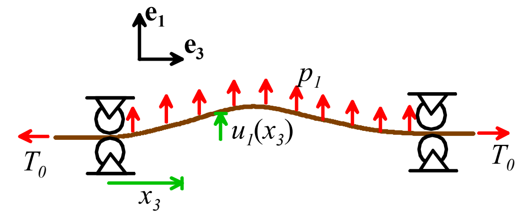
Boundary conditions:

Either: $\mathbf{u} = \mathbf{u}^* \quad x_3 = 0 \quad x_3 = L$ Or: $\mathbf{T} = -\mathbf{P}(0) \quad x_3 = 0 \quad \mathbf{T} = \mathbf{P}(L) \quad x_3 = L$

Either: $du_1/dx_3 = \theta_2^* \quad -du_2/dx_3 = \theta_1^*$ Or: $\mathbf{M} = -\mathbf{Q} \ (x_3 = 0) \quad \mathbf{M} = \mathbf{Q} \ (x_3 = L)$

# Solving problems involving strings and beams

**Static deflection of string with uniform transverse load:**

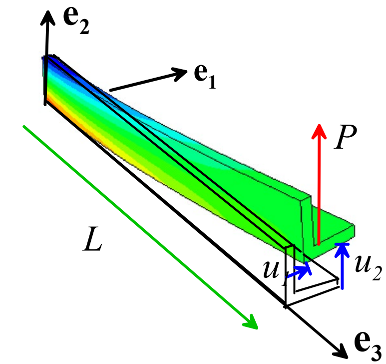



Axial Force $\frac{dT_3}{dx_3} \approx 0 \qquad T_3 = T_0 \;\; @ \;\; x_3 = 0, L \qquad \Rightarrow T_3 = T_0$

Transverse motion $\frac{d}{dx_3}\left(T_3 \frac{du_1}{dx_3}\right) + p_1 = 0 \qquad u_1 = 0 \;\; @ \;\; x_3 = 0, L$

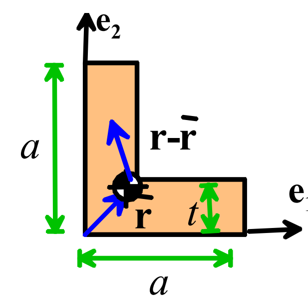
Solution: $u_1 = \frac{p_1}{2T_0} x_3 (L - x_3) \qquad T_1 = T_0 \frac{du_1}{dx_3} = \frac{p_1}{2}(L - 2x_3)$

**Static deflection of an end loaded cantilever beam with L shaped sect:**



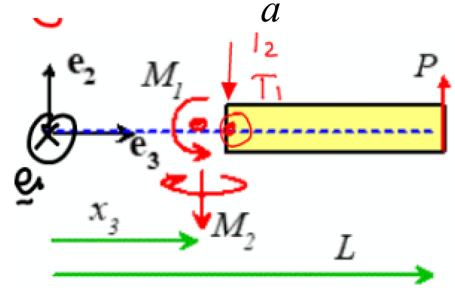

Calculate area moments of inertia $I_{11} \approx 5a^3 t / 24 \qquad I_{12} \approx -a^3 t / 24$

Calculate moments with method of sections $M_1 = -(L - x_3)P \qquad M_2 = 0$




Moment-Curvature relation $\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \frac{d^2}{dx_3^2} \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} -(L - x_3)P \\ 0 \end{bmatrix}$

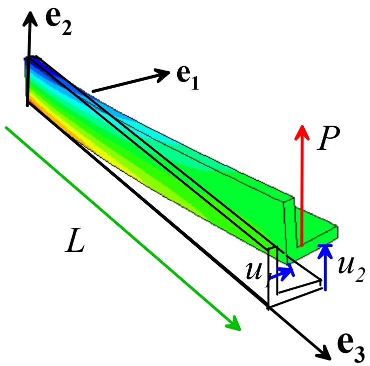
Solve $\frac{d^2 u_2}{dx_3^2} = \frac{24}{16Ea^3 t} 5P(L - x_3) \qquad \frac{d^2 u_1}{dx_3^2} = \frac{3}{5} \frac{d^2 u_2}{dx_3^2}$

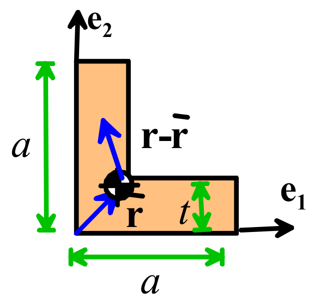



Boundary conditions $u_1 = u_2 = \frac{du_1}{dx_3} = \frac{du_2}{dx_3} = 0 \qquad x_3 = 0$

Solution $u_1 = \frac{3P}{4Ea^3 t} x_3^2 (3L - x_3) \qquad u_2 = \frac{5P}{4Ea^3 t} x_3^2 (3L - x_3)$

# Solving beam problems with MATLAB




```
syms a t EE P L x3 C1 C2 C3 C4 real
syms u1(x3) u2(x3)
II1 = a^3*t/24*[5,3;3,5]; % This is [I11,-I12;-I12;I22] - regular area moment of inertia matrix
II2 = a^3*t/24*[5,-3;-3,5]; % This is [I22,I12;I11,I22] that appears in some of the equations
uvec = [u1(x3);u2(x3)]; %vector of unknown displacements
diffeq = EE*II2*diff(uvec,x3,4) ==[0;0];  % The differential equation
BC1 = subs(uvec,x3,0)==0;  BC2 = subs(diff(uvec,x3),x3,0)==0; % BCs at x3=0 are easy
% Matlab 'dsolve' can't handle the moment/force boundary conditions so instead
% we prescribe the displacement and slope at x3=L using C1,C2,C3,C4 (unknown constants)
% and then solve for these unknown constants using the correct boundary conditions later
BC3 = subs(uvec,x3,L)==[C1;C2];  BC4 = subs(diff(uvec,x3),x3,L)==[C3;C4];
sol = dsolve(diffeq,[BC1,BC2,BC3,BC4]);
uvecsol = [sol.u1;sol.u2] % This solution still contains the unknowns C1,C2,C3,C4
mvec = -EE*II2*diff(uvecsol,x3,2); % NB: mvec contains [-M2;M1]
Tvec = -EE*II2*diff(uvecsol,x3,3); % Tvec contains [T1;T2]
eq1 = subs(mvec,x3,L)==[0;0];  % Moments are zero at x3=L
eq2 = subs(Tvec,x3,L)==[0;P];  % Internal force is [0;P] at x3=L
[C1sol,C2sol,C3sol,C4sol] = solve([eq1,eq2],[C1,C2,C3,C4]); % Solve for C1,C2,C3,C4
uvecsol = simplify(subs(uvecsol,[C1,C2,C3,C4],[C1sol,C2sol,C3sol,C4sol])) % subst back
```

uvecsol =

$$\begin{pmatrix} \frac{x_3^2\,(3\,C_1 - C_3\,L)}{L^2} - \frac{x_3^3\,(2\,C_1 - C_3\,L)}{L^3} \\ \frac{x_3^2\,(3\,C_2 - C_4\,L)}{L^2} - \frac{x_3^3\,(2\,C_2 - C_4\,L)}{L^3} \end{pmatrix}$$

uvecsol =

$$\begin{pmatrix} \frac{3\,P\,x_3^2\,(3\,L - x_3)}{4\,\mathrm{EE}\,a^3\,t} \\ \frac{5\,P\,x_3^2\,(3\,L - x_3)}{4\,\mathrm{EE}\,a^3\,t} \end{pmatrix}$$

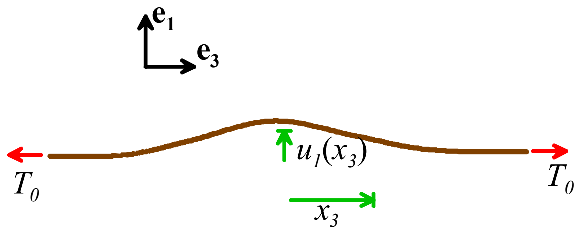
# Concept Checklist

**12. Dynamics and vibrations of elastic solids**

- Understand traveling wave motion in a string; be able to derive wave equation and solve it.
- Understand reflections of traveling waves at fixed and free ends of a string
- Be familiar with dispersive wave motion in beams
- Be familiar with P and S plane wave motion in 3D elastic solids and be able to calculate the wave speeds
- Understand reflection of P and S waves incident normal to a boundary
- Be aware of other waves in elastic solids (Rayleigh waves)

- Understand vibrational motion of strings and beams (mode shapes; natural frequencies)
- Be able to calculate natural frequencies of beams and strings

# Traveling waves on strings




Wave equation $\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2}$ $\quad c = \sqrt{\frac{T_0}{\rho A}}$

With initial conditions $u_1(0, x_3) = w_0(x_3) \quad \partial u_1 / \partial t = 0$

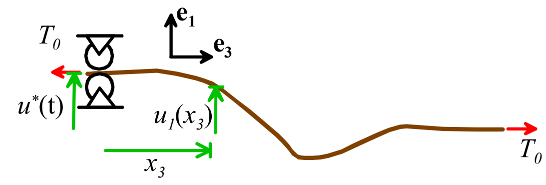
General solution to wave equation $u_1(x_3, t) = f(x_3 - ct) + g(x_3 + ct)$

Calculate *f,g* from initial conditions and/or boundary conditions

$$u_1(x_3, 0) = f(x_3) + g(x_3) = w_0$$

Initial conditions $\frac{\partial u_1(x_3, 0)}{\partial t} = c\left(-f'(x_3) + g'(x_3)\right) = 0$ $\quad f'(\lambda) = \frac{df}{d\lambda} \quad g'(\lambda) = \frac{dg}{d\lambda}$

Boundary conditions



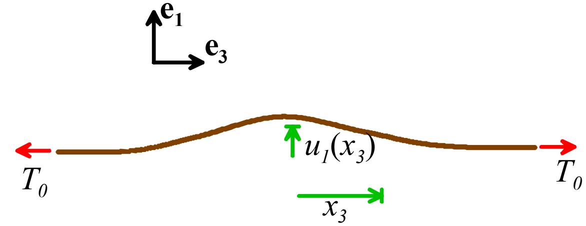

Prescribed displacement $u_1(0, t) = f(ct) + g(ct) = u^*(t)$

Free end $\frac{\partial u_1(L, t)}{\partial x_3} = \left(f'(L - ct) + g'(L + ct)\right) = 0$

# Traveling waves on strings

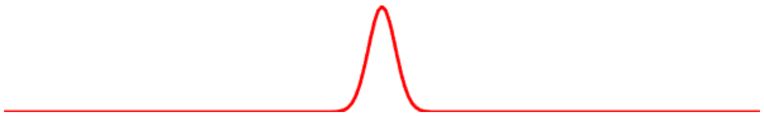
### Travelling waves on an infinite string

Wave equation $$\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} \qquad c = \sqrt{\frac{T_0}{\rho A}}$$

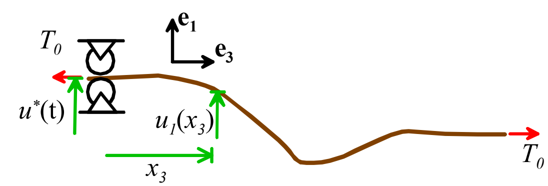



With initial conditions $u_1(0,x_3) = w_0(x_3) \quad \partial u_1 / \partial t = 0$

Solution is $u_1(t,x_3) = \left[ w_0(x_3 - ct) + w_0(x_3 + ct) \right] / 2$



### String forced at one end

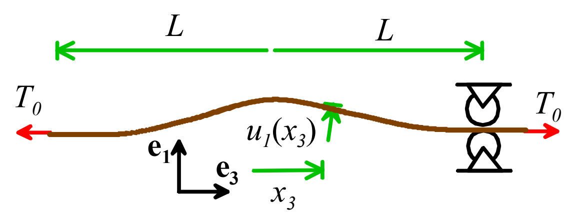



With initial conditions $u_1(0,x_3) = \partial u_1 / \partial t = 0$

Forcing $u_1(t,0) = u^*(t)$

Solution is $$u_1(t,x_3) = \begin{cases} u^*(t - x_3/c) & t > x_3/c \\ 0 & t < x_3/c \end{cases}$$

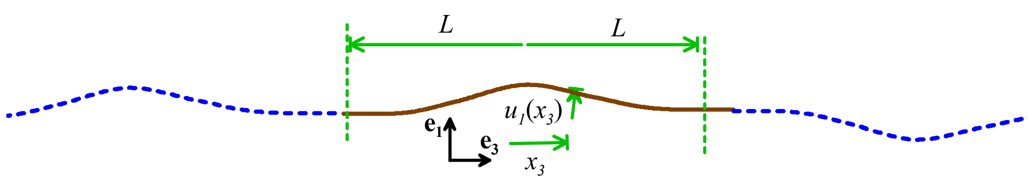

### Reflections at boundaries



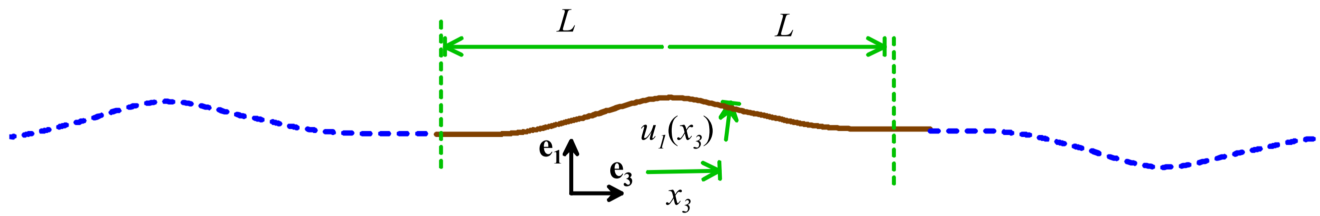

With initial conditions $u_1(0,x_3) = w_0(x_3) \quad \partial u_1 / \partial t = 0$

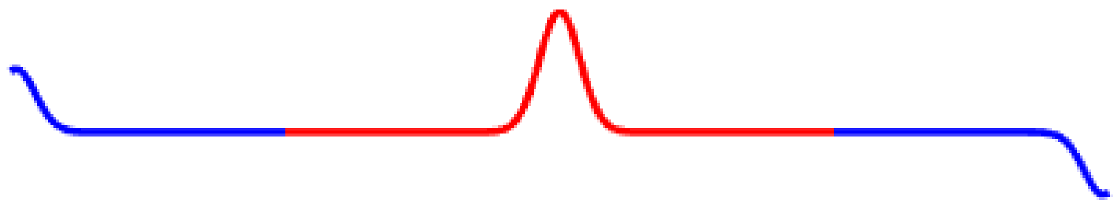
Boundary conditions $u_1(t,L) = 0 \qquad \partial u_1(t,-L) / \partial t = 0$

Solve by extending string

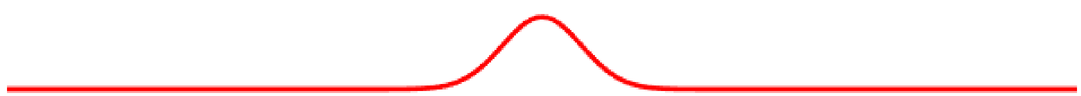


# Reflections at free and fixed ends



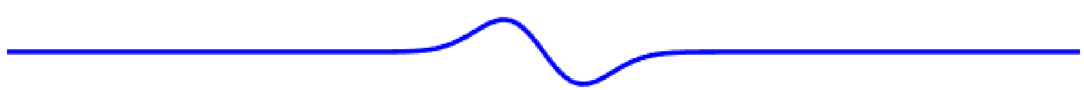



Free end



Fixed end

Displacement



Transverse force

At a fixed end:

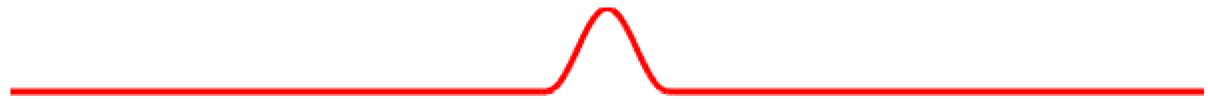
- Positive displacement reflects as negative displacement
- Positive transverse force reflects as positive transverse force

At a free end:

- Positive displacement reflects as positive displacement
- Positive transverse force reflects as negative transverse force

# Comparison of waves in beams and strings

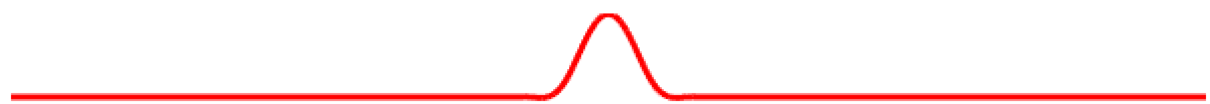
**String**



Governing equation $$\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2}$$

Has traveling wave solutions $u_1(t, x_3) = f(x_3 \pm ct)$

**Beam**



Governing equation $$\frac{d^4 u_1}{dx_3^4} + \frac{1}{\beta^2}\frac{d^2 u_1}{dt^2} = 0 \qquad \beta = \sqrt{\frac{EI}{\rho A}}$$

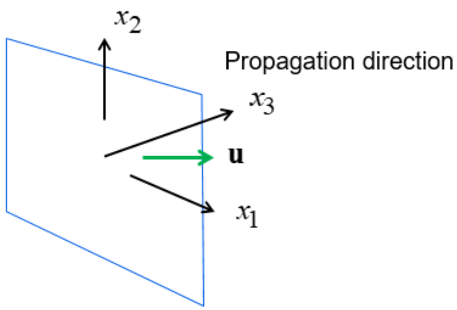
No general traveling wave solution, but harmonic motion, eg $u_1(t, x_3) = \cos[k(x_3 \pm ct)]$ $\quad c = k\beta$ satisfies EOM

Wave speed depends on wavelength (or wave number $k = 2\pi / \lambda$ ).
General disturbances contain waves with a spectrum of wave numbers – short wavelengths travel faster than long ones
This type of wave motion is called 'dispersive'

# Plane waves in 3D elastic solids

**Plane waves in large elastic solids:**

1. Deformation $\mathbf{u}(x_3,t)=u_1(x_3,t)\mathbf{e}_1+u_3(x_3,t)\mathbf{e}_3$
2. Governing equations




$$\frac{\partial^2 u_1}{\partial x_3^2}=\frac{1}{c_s^2}\frac{\partial^2 u_1}{\partial t^2} \qquad \frac{\partial^2 u_3}{\partial x_3^2}=\frac{1}{c_L^2}\frac{\partial^2 u_3}{\partial t^2}$$

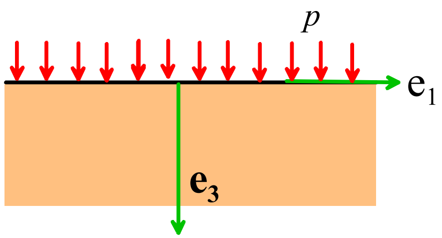
$$\sigma_{13}=\sigma_{31}=\frac{E}{2(1+\nu)}\frac{\partial u_1}{\partial x_3}=\rho c_s^2\frac{\partial u_1}{\partial x_3} \qquad \sigma_{33}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial u_3}{\partial x_3}=\rho c_L^2\frac{\partial u_3}{\partial x_3}$$

3. Wave speeds

$$c_s=\sqrt{\frac{E}{2(1+\nu)\rho}}$$ Shear (S) wave

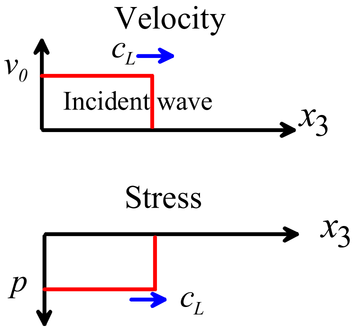
$$c_L=\sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}$$ Pressure (P) wave

**Example:** A large elastic solid is at rest for *t<0*. For time *t>0* its surface is subjected to a constant uniform pressure *p*. Calculate the stress and velocity distribution in the solid.




Solution is a P wave $$u_3=\frac{p}{\rho c_L}\langle t-x_3/c_L\rangle \qquad \langle x\rangle=\begin{cases}x & x>0\\ 0 & x<0\end{cases}$$

$$v_3=\frac{\partial u_3}{\partial t}=\frac{p}{\rho c_L}H(t-x_3/c_L) \qquad \sigma_{33}=-pH(t-x_3/c_L) \qquad H(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases}$$




(S wave would be generated by shear stress on surface)

# Rayleigh wave

Special wave that propagates at the surface of an elastic solid

Wave speed satisfies $$\left(2-\frac{c_R^2}{c_s^2}\right)^2-4\left(1-\frac{(1-2\nu)}{2(1-\nu)}\frac{c_R^2}{c_s^2}\right)^{1/2}\left(1-\frac{c_R^2}{c_s^2}\right)^{1/2}=0$$

## Displacement

$$u_1=\frac{U_0 ik}{(k^2-\beta_T^2)\beta_L}\exp\left(ik(x_1-c_Rt)\right)\left\{(k^2+\beta_T^2)\exp\left(-\beta_L x_2\right)-2\beta_L\beta_T\exp\left(-\beta_T x_2\right)\right\}$$

$$u_2=\frac{U_0}{(k^2-\beta_T^2)}\exp\left(ik(x_1-c_Rt)\right)\left\{2k^2\exp\left(-\beta_T x_2\right)-(k^2+\beta_T^2)\exp\left(-\beta_L x_2\right)\right\}$$

$$\beta_L=k\sqrt{1-c_R^2/c_L^2} \qquad \beta_T=k\sqrt{1-c_R^2/c_s^2}$$

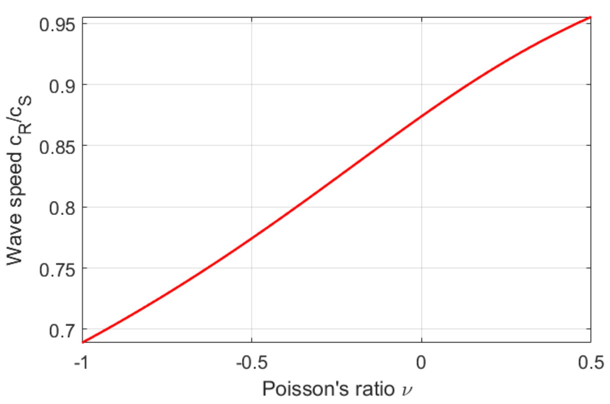
Wave number $k=2\pi/\lambda$

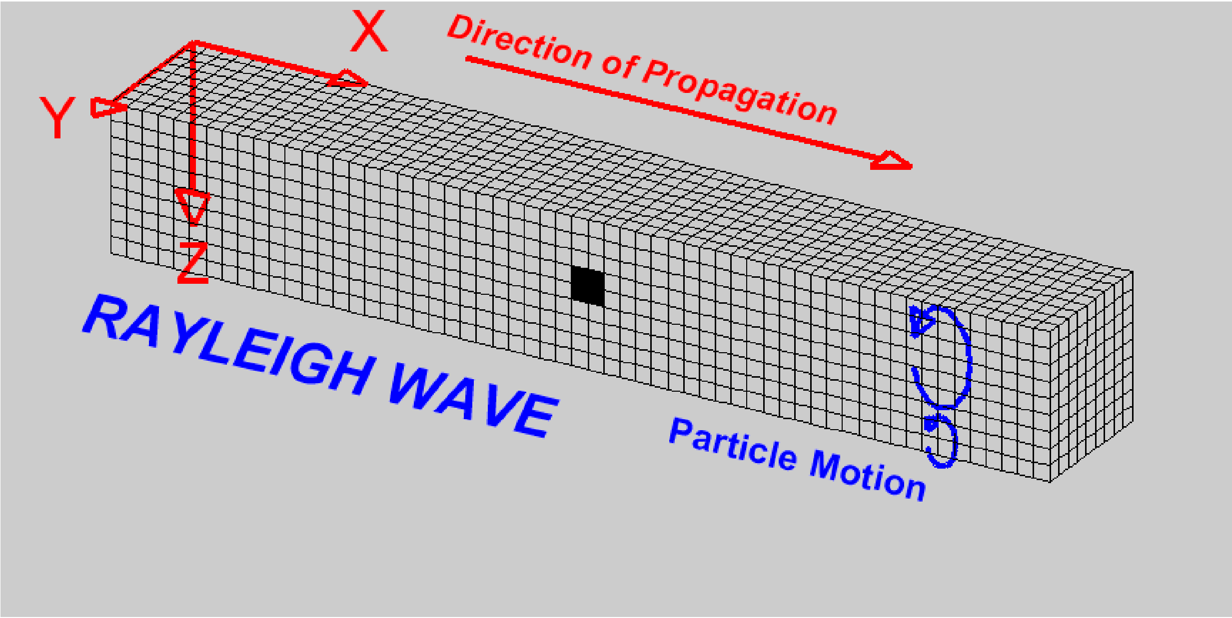
## Stress

$$\sigma_{11}=\frac{U_0E\exp\left(ik(x_1-c_Rt)\right)}{(k^2-\beta_T^2)(1+\nu)(1-2\nu)\beta_L}\left\{k^2\left[\nu(\beta_L^2+\beta_T^2)-(1-\nu)(k^2+\beta_T^2)\right]\exp\left(-\beta_L x_2\right)+2k^2\beta_T\beta_L(1-2\nu)\exp\left(-\beta_T x_2\right)\right\}$$

$$\sigma_{22}=\frac{U_0E\exp\left(ik(x_1-c_Rt)\right)}{(k^2-\beta_T^2)(1+\nu)(1-2\nu)\beta_L}\left\{(k^2+\beta_T^2)\left[(1-\nu)\beta_L^2-\nu k^2\right]\exp\left(-\beta_L x_2\right)-2k^2\beta_T\beta_L(1-2\nu)\exp\left(-\beta_T x_2\right)\right\}$$

$$\sigma_{12}=\frac{iU_0kE(k^2+\beta_T^2)}{(k^2-\beta_T^2)(1+\nu)}\exp\left(ik(x_1-c_Rt)\right)\left\{\exp\left(-\beta_T x_2\right)-\exp\left(-\beta_L x_2\right)\right\}$$



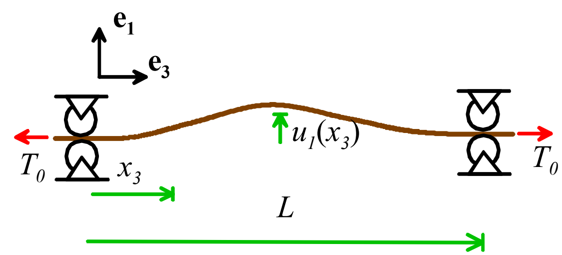
# Vibrations in elastic solids

## Natural frequencies and mode shapes

Elastic solids will exhibit simple harmonic motion when released from rest from a special initial displacement field

The special initial displacements are called 'mode shapes'

The special frequencies are called 'natural frequencies'




## Calculating natural frequencies

Start with governing equation (eg wave equation or beam equation)

$$\frac{\partial^2 u_3}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_3}{\partial t^2}$$

Guess harmonic solution $u_3 = \cos(\omega t + \phi) f(x_3)$

Governing equation gives ODE for *f* $\left(\frac{\partial^2 f}{\partial x_3^2} + \frac{\omega^2}{c^2} f\right)\cos(\omega t + \phi) = 0$

General solution eg $u_3 = A\sin kx_3 + B\cos kx_3 \qquad k = \omega / c$

Write boundary conditions in form $[H]\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Set $\det\left([H]\right) = 0$ and solve for *k*. Then $ck = \omega$ (Always multiple solutions)

Substitute *k* back into [H] to find *A*,*B* and hence find mode shape for each *k*

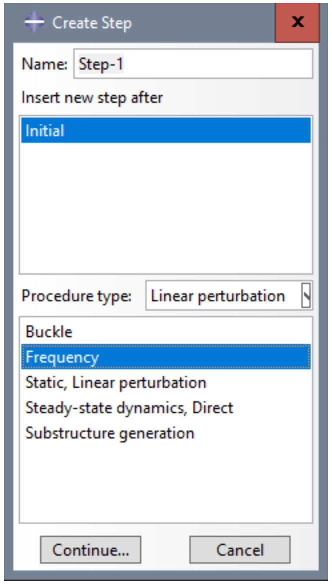
First mode

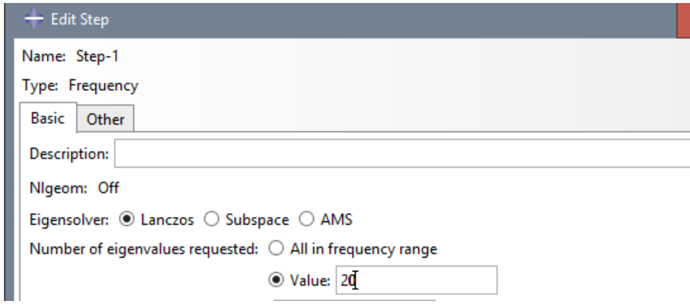
Second mode

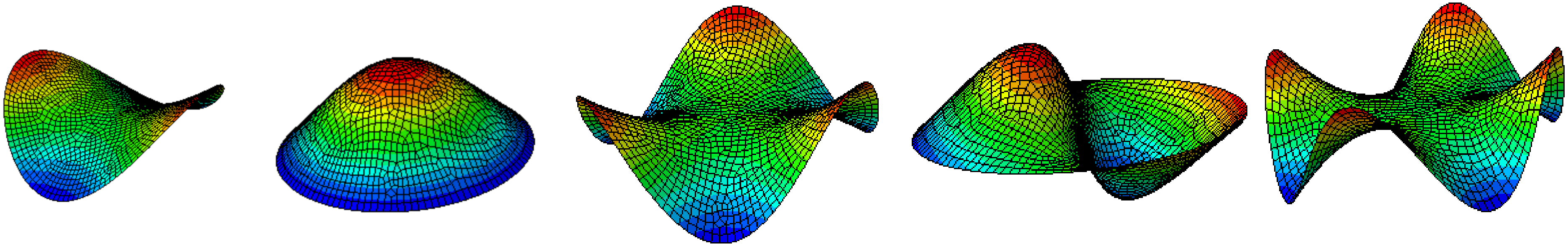
Third mode

# Calculating natural frequencies with ABAQUS

1. Set up geometry, properties, section, etc in usual way. Be sure to define density for the material
2. Create part instance in assembly in usual way
3. (Optional) – conduct a static step to pre-load structure (needed, eg, for a membrane)
4. Create a new step after optional static step, then in Step menu select ‘Linear Perturbation’ procedure, and select ‘Frequency’. Can select # vibration modes
5. Apply boundary conditions in usual way
6. Mesh solid – be careful with element choice (usually best to avoid reduced integration/incompatible modes as they have artificial deformation modes; also if elements will lock that will cause serious problems). Large # vibration modes will require fine mesh.
7. Run job in usual way
8. Frequencies, mode shapes are displayed in Visualization Module



# Concept Checklist

**13. Plasticity**

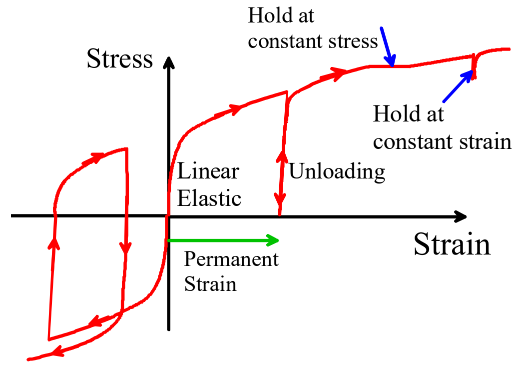
- Be familiar with the qualitative behavior of metallic materials loaded beyond yield, including:
  - Yielding; permanent deformation; creep; cyclic plasticity in uniaxial tension/compression
  - Insensitivity of yield to hydrostatic pressure; plastic strains are volume preserving;
  - Levy-Mises flow law.
- Understand the difference between rate dependent and rate independent models of plasticity
- Understand the equations used in rate independent isotropic hardening plasticity models, including:
  - The yield criterion
  - Partitioning of strain into elastic and plastic parts
  - Strain hardening relations
  - The plastic flow law
- Be able to calculate plastic strains resulting from simple cycles of stress (uniaxial, biaxial)
- Understand the concept of plastic collapse in an elastic-plastic solid
- Be able to calculate plastic collapse loads in axially and spherically symmetric solids

# Plasticity

## Behavior of metals/polymers loaded beyond yield

Different plasticity models exist for:

- Room T quasi-static loading (rate independent/isotropic)
- High temperatures (creep) (rate dependent viscoplasticity)
- Dynamic loading (rate dependent viscoplasticity)
- Cyclic loading (kinematic hardening)
- Special models exist for soils/single xtals/polymers

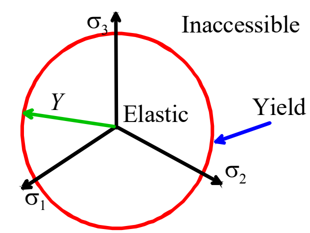



## Rate Independent Plasticity Model

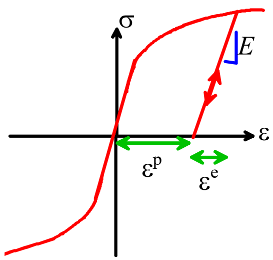
Yield Criterion (Von Mises) $\sigma_e < Y$ for elastic behavior

$$\sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}} = \sqrt{\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right\}}$$

$$S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$$




Von-Mises yield surface



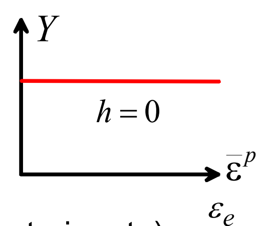

Strain Partitioning

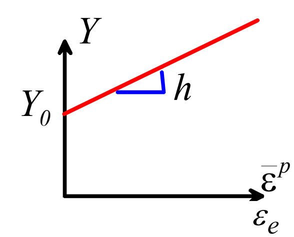
$$\frac{d\varepsilon_{ij}}{dt} = \frac{d\varepsilon_{ij}^e}{dt} + \frac{d\varepsilon_{ij}^T}{dt} + \frac{d\varepsilon_{ij}^p}{dt}$$

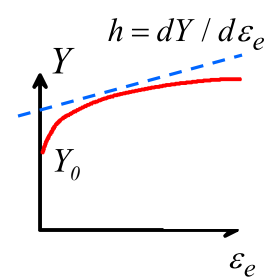
Hardening Rule $Y(\varepsilon_e)$

$$\frac{d\varepsilon_e}{dt} = \sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^p}{dt}\frac{d\varepsilon_{ij}^p}{dt}} = \frac{3}{2h\sigma_e}\left\langle S_{kl}\frac{d\sigma_{kl}}{dt}\right\rangle$$

(Von Mises plastic strain rate)



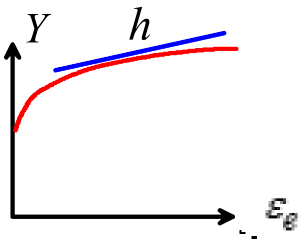

Flow Rule

$$\frac{d\varepsilon_{ij}^p}{dt} = \frac{d\varepsilon_e}{dt}\frac{3}{2}\frac{S_{ij}}{\sigma_e}$$

$$\langle x\rangle = \begin{cases} x & x>0 \\ 0 & x<0 \end{cases}$$

# Summary of elastic-plastic stress-strain relations

$$\frac{d\varepsilon_{ij}}{dt}=\begin{cases}\dfrac{1+\nu}{E}\left(\dfrac{d\sigma_{ij}}{dt}-\dfrac{\nu}{1+\nu}\dfrac{d\sigma_{kk}}{dt}\delta_{ij}\right)+\alpha\dfrac{d\Delta T}{dt}\delta_{ij} & \sigma_e-Y(\varepsilon_e)<0 \quad \text{(Elastic)}\\[2ex] \dfrac{1+\nu}{E}\left(\dfrac{d\sigma_{ij}}{dt}-\dfrac{\nu}{1+\nu}\dfrac{d\sigma_{kk}}{dt}\delta_{ij}\right)+\alpha\dfrac{d\Delta T}{dt}\delta_{ij}+\dfrac{1}{h}\dfrac{3}{2}\dfrac{\left\langle S_{kl}\dfrac{d\sigma_{kl}}{dt}\right\rangle}{\sigma_e}\dfrac{3}{2}\dfrac{S_{ij}}{\sigma_e} & \sigma_e-Y(\varepsilon_e)=0 \quad \text{(Plastic)}\end{cases}$$




$d\varepsilon_{ij}^{e}/dt$ (elastic part) $\qquad d\varepsilon_{ij}^{p}/dt$ (plastic part)

$$\sigma_e=\sqrt{\frac{3}{2}S_{ij}S_{ij}} \qquad S_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij} \qquad \frac{d\varepsilon_e}{dt}=\sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^{p}}{dt}\frac{d\varepsilon_{ij}^{p}}{dt}}=\frac{3}{2h\sigma_e}\left\langle S_{kl}\frac{d\sigma_{kl}}{dt}\right\rangle \qquad \langle x\rangle=\begin{cases}x & x>0\\ 0 & x<0\end{cases}$$

## Matrix Form

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}=\begin{cases}\dfrac{1}{E}\begin{bmatrix}1&-\nu&-\nu&0&0&0\\ -\nu&1&-\nu&0&0&0\\ -\nu&-\nu&1&0&0&0\\ 0&0&0&2(1+\nu)&0&0\\ 0&0&0&0&2(1+\nu)&0\\ 0&0&0&0&0&2(1+\nu)\end{bmatrix}\dfrac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\dfrac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix} & \sigma_e-Y(\varepsilon_e)<0 \quad \text{(Elastic)}\\[2ex] \dfrac{1}{E}\begin{bmatrix}1&-\nu&-\nu&0&0&0\\ -\nu&1&-\nu&0&0&0\\ -\nu&-\nu&1&0&0&0\\ 0&0&0&2(1+\nu)&0&0\\ 0&0&0&0&2(1+\nu)&0\\ 0&0&0&0&0&2(1+\nu)\end{bmatrix}\dfrac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\dfrac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}+\dfrac{3}{2h\sigma_e}\left\langle S_{kl}\dfrac{d\sigma_{kl}}{dt}\right\rangle\dfrac{3}{2}\begin{bmatrix}S_{11}/\sigma_e\\ S_{22}/\sigma_e\\ S_{33}/\sigma_e\\ 2S_{23}/\sigma_e\\ 2S_{13}/\sigma_e\\ 2S_{12}/\sigma_e\end{bmatrix} & \sigma_e-Y(\varepsilon_e)=0 \quad \text{(Plastic)}\end{cases}$$

# Concept Checklist

**14. Failure**

- Understand that solids may fail by (1) geometric instability (elastic buckling; necking or localization) or (2) by material failure (fracture, ductile rupture, or fatigue)
- Be able to calculate buckling loads for elastic slender members under axial loading
- Be familiar with the features of brittle and ductile failure in solids subjected to monotonic loading
- Understand stress-based isotropic and anisotropic failure criteria for brittle materials

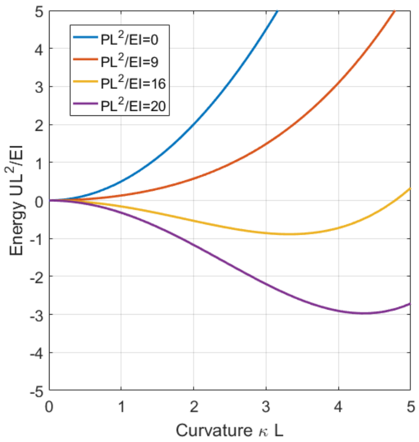
- Be familiar with failure by necking in plastically deforming specimens subjected to uniaxial loading
- Be able to calculate the critical strain at the onset of necking in a specimen subjected to uniaxial tension
- Be familiar with necking failures in sheets and understand the concept of a forming limit diagram
- Be familiar with strain based criteria for material failure by void nucleation and growth

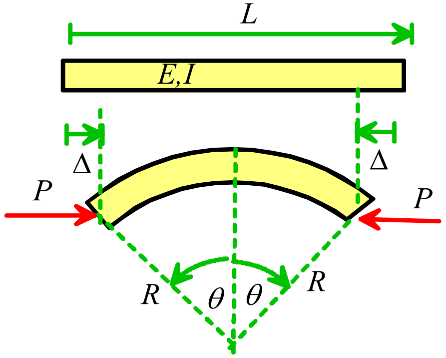
- Understand the difference between high cycle and low cycle fatigue
- Be familiar with stress and strain based criteria for fatigue failure under cyclic loading

# Failure by geometric instability in elastic solids (buckling)

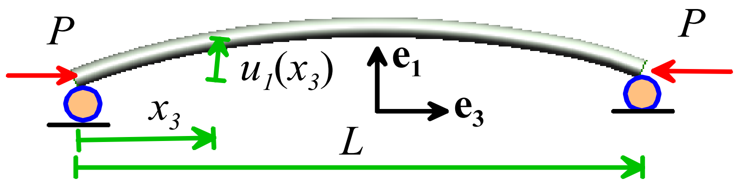
## Simple explanation for buckling

1. Below buckling load potential energy is minimized by straight beam
2. Above buckling load, potential energy is minimized by bent beam




## Calculating buckling loads in columns




Beam equation (2D, static, with $I_{12}=0$ )

$$EI\frac{d^4u_1}{dx_3^4}=\frac{d^2u_1}{dx_3^2}T_3+p_1 \qquad \frac{dT_3}{dx_3}+p_3\approx 0$$

With no transverse loads and constant axial force $p_1=0,\quad T_3=-P \Rightarrow EI\frac{d^4u_1}{dx_3^4}+P\frac{d^2u_1}{dx_3^2}=0$

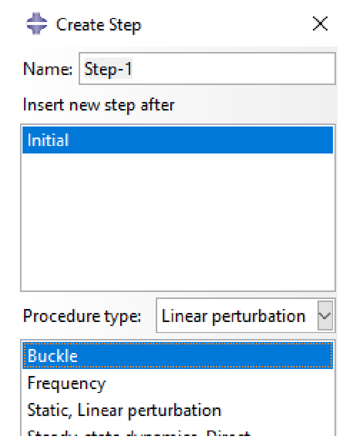
General solution $u_1=A\sin kx_3+B\cos kx_3+Cx_3+D$ with $k^2=\frac{P}{EI}$

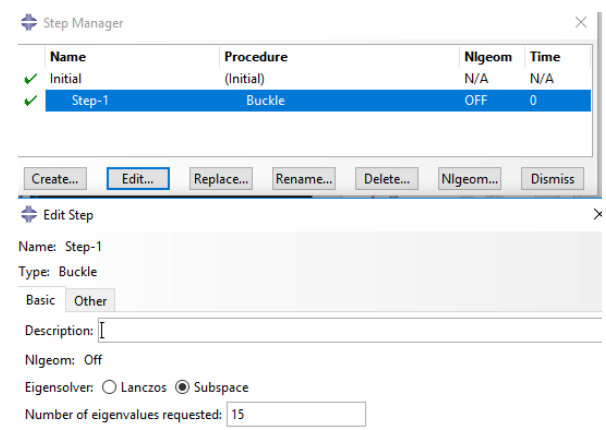
Write boundary conditions at ends in the form $[H]\begin{bmatrix}A\\B\\C\\D\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

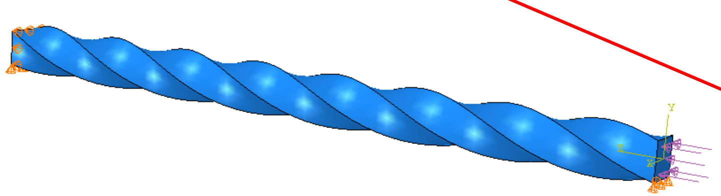
Set $\det([H])=0$ and solve for *k*. Then $EIk^2=P_{crit}$ (always many solutions – use the lowest one)

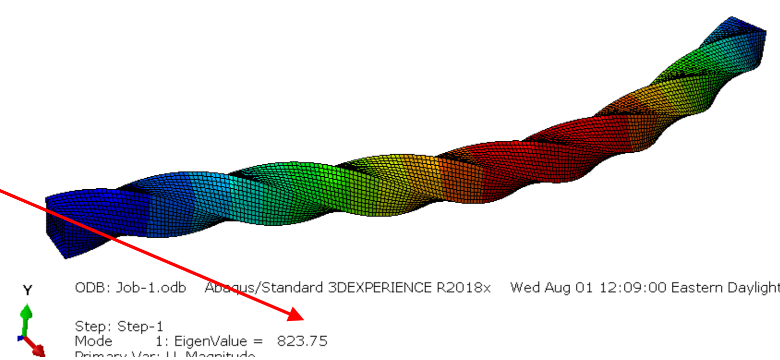
# Calculating buckling loads with ABAQUS

1. Set up geometry, properties, section, etc in usual way.
2. Create part instance in assembly in usual way
3. Create a new step after optional static step, then in Step menu select ‘Linear Perturbation’ procedure, and select ‘Buckle’. Can select # buckling modes
4. Apply boundary conditions in usual way. Be sure to include a load that will cause buckling. The load can have an arbitrary magnitude – ABAQUS will compute how much the load needs to be multiplied by to cause buckling.
5. Mesh solid – be careful with element choice (usually best to avoid reduced integration/incompatible modes as they have artificial deformation modes; also if elements will lock that will cause serious problems). Large # buckling modes will require fine mesh. Experiment with different element types.
6. Run job in usual way
7. Bucklin mode shapes are displayed in Visualization Module. The ‘Eigenvalue’ is the scale factor applied to the loads



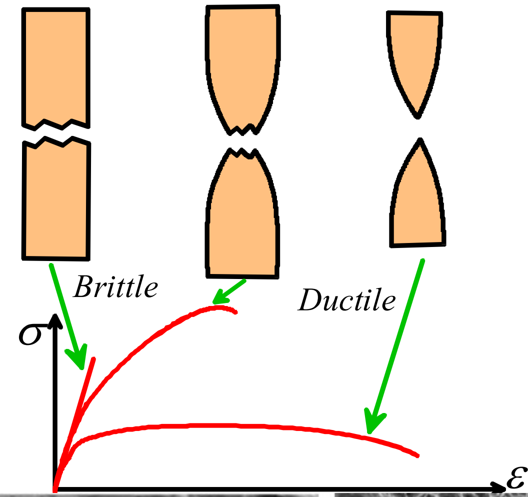



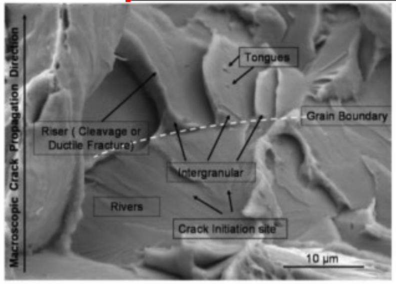





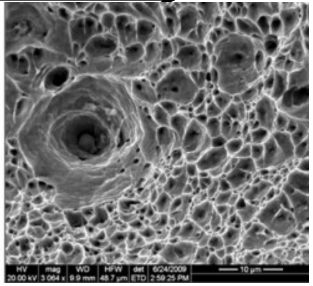
## Failure under monotonic loading

1. Brittle failure – little permanent deformation prior to failure; faceted failure surface
2. Ductile failure – extensive permanent deformation prior to failure; dimpled fracture surface




Cleavage fracture (brittle)



Dimpled rupture (ductile)

## Brittle failure criteria

$\{\sigma_1, \sigma_2, \sigma_3\}$ Principal stresses $\sigma_1 > \sigma_2 > \sigma_3$

$\tau = (\sigma_1 - \sigma_3)/2 \qquad \sigma_m = (\sigma_1 + \sigma_3)/2$

### Isotropic failure criteria

$\sigma_1 > 0$ Failure by fracture when $\sigma_1 = \sigma_f$

$\sigma_1 < 0$ Failure by crushing (eg Mohr Coulomb criterion)

$\tau + \sigma_m \sin\phi - c\cos\phi < 0$ Safe

$\tau + \sigma_m \sin\phi - c\cos\phi = 0$ Fails

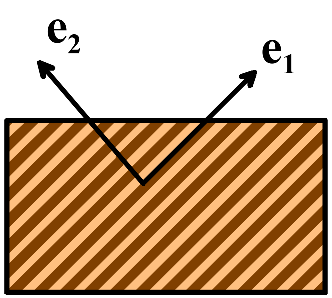
### Anisotropic failure criteria (eg Tsai-Hill for composite laminates)

Basis parallel/perpendicular to fibers $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

Plane stress state $\{\sigma_{11}, \sigma_{22}, \sigma_{12}\}$

Failure criterion

$$\left(\frac{\sigma_{11}}{\sigma_{TS1}}\right)^2 + \left(\frac{\sigma_{22}}{\sigma_{TS2}}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{\sigma_{TS1}} + \left(\frac{\sigma_{12}}{\sigma_{SS}}\right)^2 = 1$$



# Failure by geometric instability in plastic materials

## Failure by necking in tension

Specimen starts to neck when load-displacement curve decreases (loss of x-sect area weakens specimen)

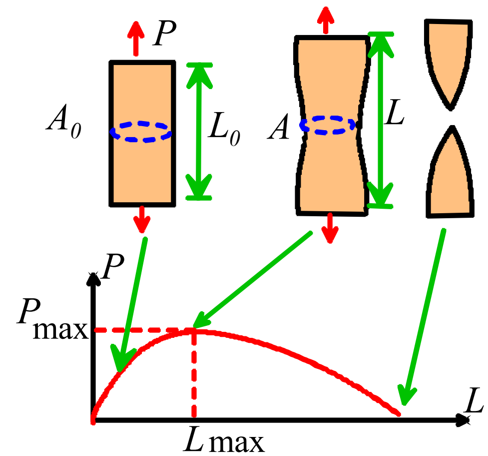
$$P = AY(\varepsilon)$$

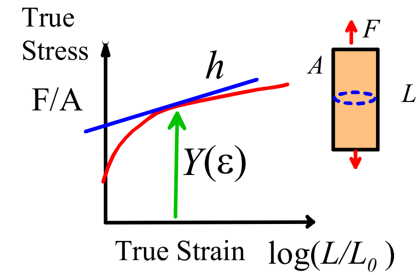
$$\frac{dP}{dL} = \frac{dA}{dL}Y + A\frac{dY}{d\varepsilon}\frac{d\varepsilon}{dL}$$

Incompressibility $AL = const$

True strain $\varepsilon = \log(L / L_0)$

**Combine** $\frac{dY}{d\varepsilon} = Y$ (Considere criterion)

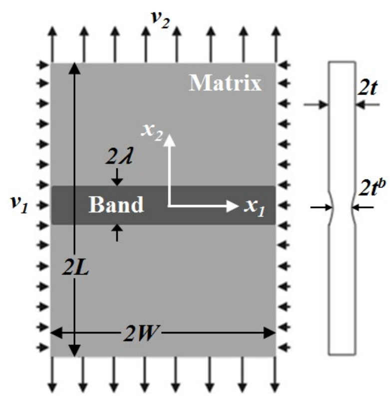



## Necking in sheet materials – forming limit diagrams

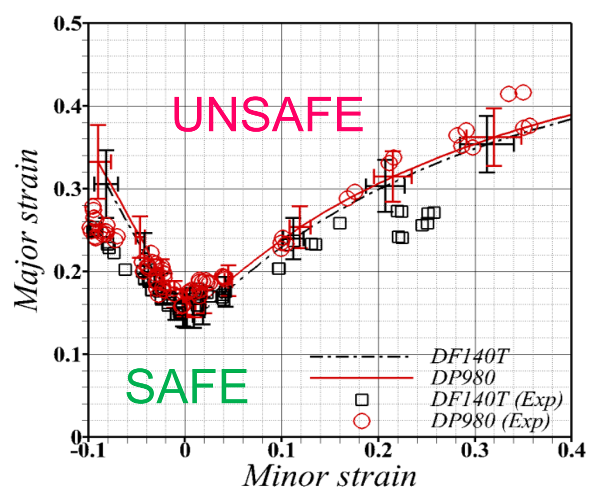
Sheet necks when strain in sheet reaches a critical value (depends on ratio of major to minor strain, as well as flow stress and hardening rate of material, and thickness variations in sheet)

Use FLD as a failure criterion




Major Strain $\varepsilon_{22}$

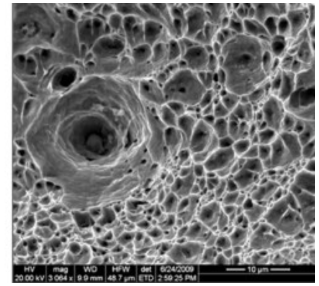
Minor Strain $\varepsilon_{11}$

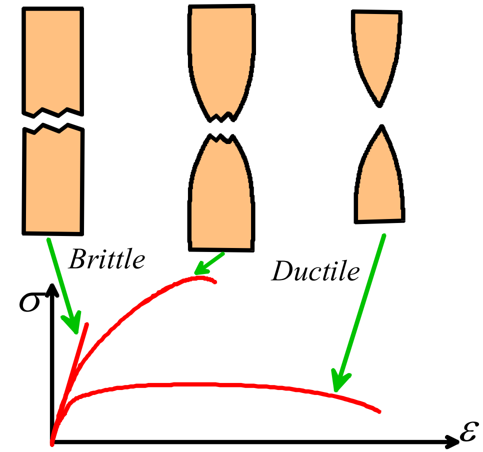


# Plastic Strain based ductile failure criteria

## Failure mechanism

- Ductile materials usually fail because voids nucleate at 2nd phase particles, grow, and coalesce.
- Voids nucleate at a critical strain
- They grow faster when subjected to tensile hydrostatic stress






## Simple plastic strain based criterion

Von Mises Plastic Strain Rate $$\frac{d\varepsilon_e}{dt} = \sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^p}{dt}\frac{d\varepsilon_{ij}^p}{dt}}$$

Total Von-Mises Strain $$\varepsilon_e = \int_0^t \frac{d\varepsilon_e}{dt}dt$$

Failure criterion

$\varepsilon_e < \varepsilon_f$ (safe)

$\varepsilon_e = \varepsilon_f$ (Fail)

This criterion works OK when hydrostatic stress is low

## Johnson-Cook Criterion

Hydrostatic stress $\sigma_m = \sigma_{kk} / 3$ Von Mises stress $\sigma_e = \sqrt{\frac{3}{2}S_{ij}S_{ij}}$

Damage criterion $$D = \int_0^t \frac{\frac{d\varepsilon_e}{dt}}{(d_1 + d_2\exp[d_3\sigma_m / \sigma_e])(1 + d_4\log[(d\varepsilon_e / dt)/\dot{\varepsilon}_0])}dt$$

$d_1, d_2, d_3, d_4, \dot{\varepsilon}_0$ Material props

$D < 1$ (safe)

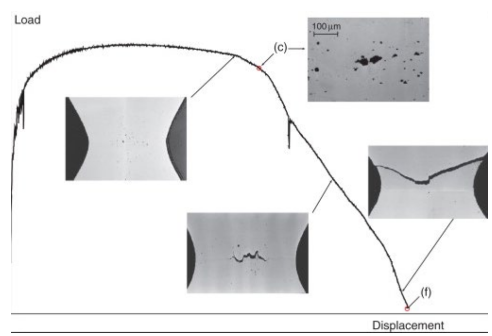
$D = 1$ (Fail)

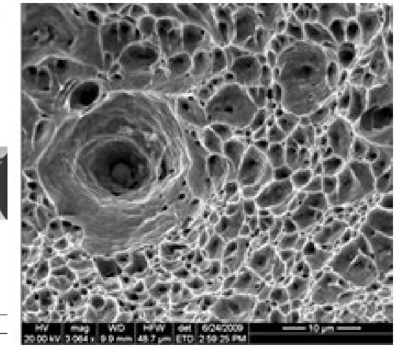
This criterion is often used for higher hydrostatic stress, as well as elevated strain rates such as machining, crash or ballistic penetration

# Porous plasticity ‘Gurson’ model in ABAQUS

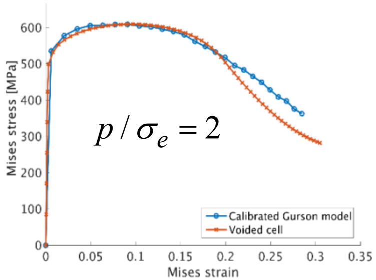
## Failure mechanism in ductile metals

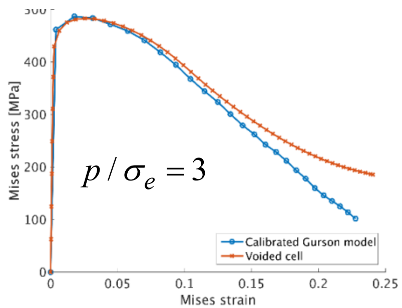
1. Specimen necks
2. Hydrostatic (tensile) stress increases in neck
3. Voids nucleate at 2nd phase particles
4. Voids grow
5. Voids coalesce to form crack

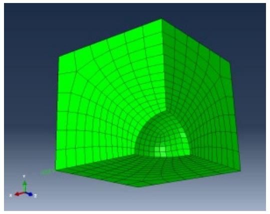


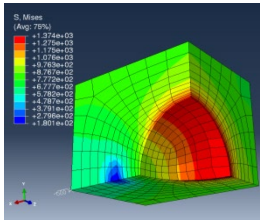



## Simulations of void growth






## Gurson (porous plasticity model)

Mises stress — Void volume fraction — Hydrostatic stress — Yield stress

Plastic strain rate magnitude
$$\dot{\varepsilon}_e = g(\sigma_e, p, \sigma_0, f^*) = \dot{\varepsilon}_0 \left[ \left( \frac{\sigma_e}{Y(\varepsilon_e)} \right)^2 + 2 q_1 f^* \cosh\left( q_2 \frac{3p}{2Y(\varepsilon_e)} \right) - (q_3 f^*)^2 - 1 \right]^{m/2}$$

Void growth and nucleation
$$\dot{f} = (1-f)\dot{\varepsilon}^p_{kk} + N_v \dot{\varepsilon}_e$$

Void nucleation rate
$$N_v = \begin{cases} \dfrac{f_N}{s_N\sqrt{2\pi}} \exp\left( -\dfrac{1}{2}\left[ \dfrac{\varepsilon^p_m - \varepsilon_N}{s_N} \right] \right) \dot{\varepsilon}_e & p > 0 \\ 0 & p < 0 \end{cases}$$

Void coalescence
$$f^* = \begin{cases} f & f < f_c \\ f_c + (1/q_1 - f_c)(f - f_c)/(f_F - f_c) & f \ge f_c \end{cases}$$

Element deleted when $f = f_F$

## Material Properties

Yield and hardening $Y(\varepsilon_e)$

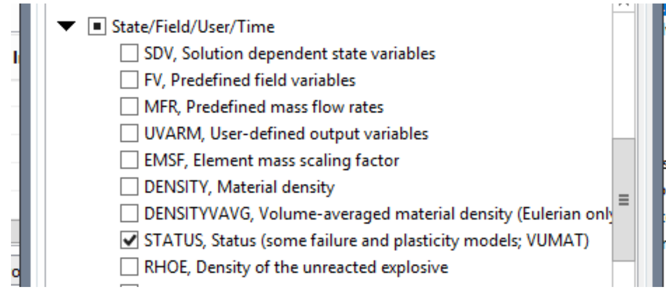
Void growth rate $q_1, q_2, q_3$

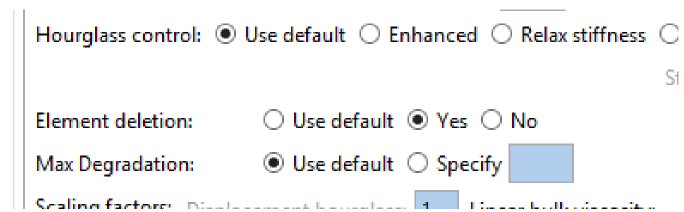
Void coalescence $f_c, f_F$

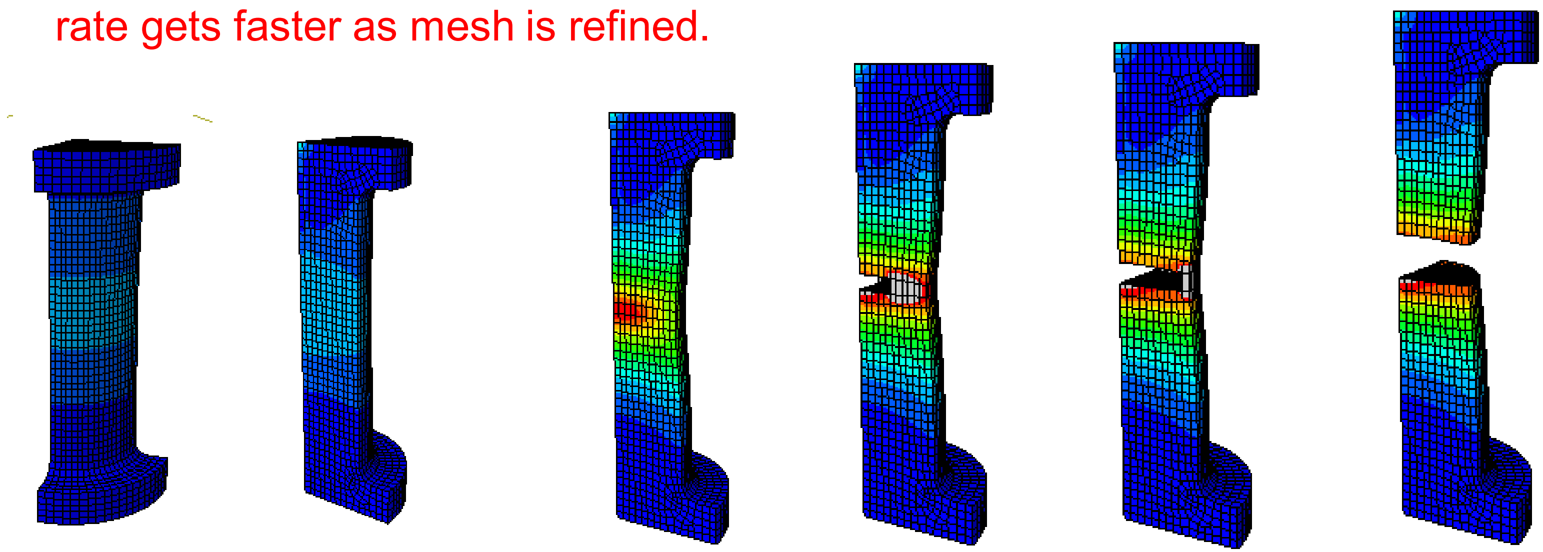
Void nucleation rate $f_N, s_N, \varepsilon_N$

# Running ABAQUS/Explicit with Gurson Model

1. Set up geometry, section, etc in usual way.
2. In the Material module select Porous Metal Plasticity
3. Enter values for q1,q2,q3 (can use 1 for each)
4. Use the Suboptions button to define $f_N, s_N, \varepsilon_N \quad f_c, f_F$
5. Define the initial value of r=1-f in the 'Relative Density'
6. Create part instance in assembly in usual way
7. Create an explicit dynamic step. To enable element deletion, use Results->Field Output and in the dialog make sure the 'Status' option is checked
8. Apply boundary conditions in usual way
9. Mesh solid – in the Element Type menu check the box for 'Element Deletion'
10. Run job in usual way
11. WARNING: Simulations with models like this are always mesh sensitive once material starts to soften – softening rate gets faster as mesh is refined.



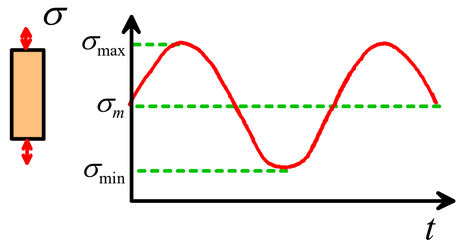



# Failure under cyclic loading

## General regimes of failure in cyclic uniaxial tension/compression

1. Low Cycle Fatigue – stresses exceed yield; failure in less than 10000 cycles; controlled by plastic strain amplitude
2. High Cycle Fatigue – stresses are below yield; failure in more than 10000 cycles; controlled by stress amplitude and mean stress

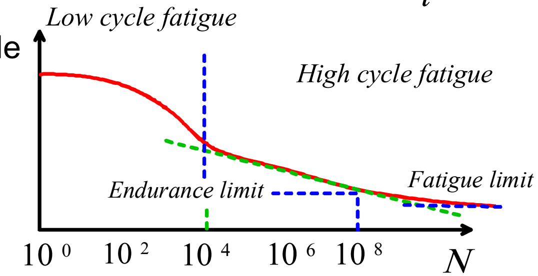



### High-cycle fatigue failure

$$\sigma_a = (\sigma_{max} - \sigma_{min})/2 \qquad \sigma_m = (\sigma_{max} + \sigma_{min})/2$$

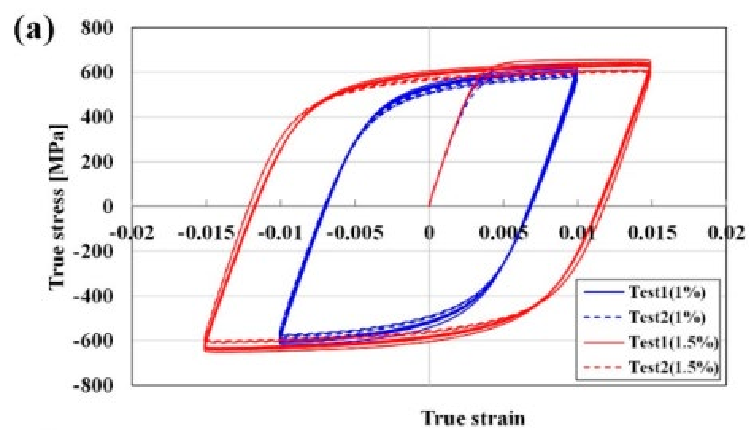
Zero mean stress: Basquin's Law $\sigma_a N^b = C$

$b, C$ are constants – depend on material, surface finish, environment




Nonzero mean stress: Goodman's rule $\sigma_a N^b = C(1 - \sigma_m / \sigma_{UTS})$

$\sigma_{UTS}$ is tensile strength




### Low-cycle fatigue failure

Coffin-Manson law $\Delta \varepsilon^p N^a = B$

$a, B$ are constants – depend on material, surface finish, environment

# Some topics we did not have time to cover ☹

**1. Elasticity theory**

- Complex variable methods, Potential formulations for 3D problems, Fourier transform techniques
- More energy methods; reciprocal theorem; complementary energy
- Anisotropic elasticity (especially important for composites and single crystals)
- More general stress waves

**2. Material models**

- Viscoelasticity
- Hyperelasticity (elastomers)
- Models of plasticity – crystal plasticity, soils, granular materials; models intended for cyclic loading
- Models for interfaces and contacts

**3. Fracture mechanics**

- Crack tip fields in elastic solids, stress intensity factors
- Energy methods for fracture – energy release rate, J integral
- Crack growth in ductile materials
- Crack growth under cyclic loading – fracture mechanics based design against fatigue failures

**4. Contact mechanics** (Stresses near contacts; compliance of a contact, fatigue/wear/fretting failures)

**5. Defect mechanics**

- Solutions and behavior of dislocations, inclusions, cracks
- Use of atomistic simulations

**6. Advanced FEA coding**

- Advanced elements
- Nonlinear materials, large geometry changes
- Explicit/implicit dynamics; modal time integration
- Contact
- Coupled solid/fluid problems, arbitrary Lagrangean/Eulerean formulations, etc