Course Outline

1. Fundamental Postulates of Solid Mechanics
2. Introduction to FEA using ABAQUS
3. Math Review, introduction to tensors and index notation
4. Describing Deformations
5. Describing Forces
6. Equations of Motion
7. Overview of material models – relating stress to strain
8. Linear Elastic Stress-Strain Relations
9. Analytical Solutions for Linear Elastic Solids
10. Energy Methods for Linear Elastic Solids
11. Implementing the Finite Element Method for Elastic Solids
12. Solids with special shapes – beams and plates
13. Dynamic elasticity – waves and vibrations
14. Plastic stress-strain relations;
15. Solutions for elastic-plastic solids
16. Modeling failure

Exam Topics

Concept Checklist

2. FEA analysis

- Be able to idealize a solid component as a 3D continuum, rod, shell or plate
- Understand how to choose a material model for a component or structure
- Be familiar with features of a finite element mesh; be able to design an suitable mesh for a component
- Understand the role of the FE mesh as a way to interpolate displacement fields
- Understand the difference between solid, shell and beam elements
- Understand that selecting inappropriate element types and poor mesh design may lead to inaccurate results
- Understand how to select boundary conditions and loading applied to a mesh
- Understand that for static analysis boundary conditions must prevent rigid motion to ensure that FEA will converge
- Understand use of tie constraints to bond meshes or to bond a rigid surface to a part
- Be able to analyze contact between deformable solids
- Be able to choose a static, explicit dynamic, or implicit dynamic analysis;

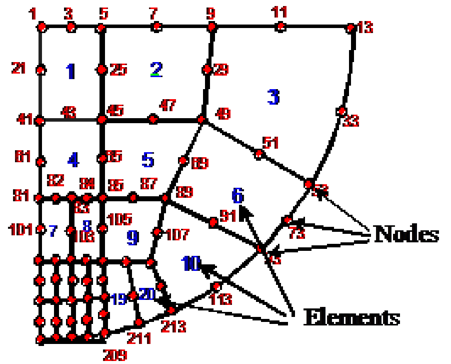
- Be able to interpret and draw conclusions from analysis predictions; have the physical insight to recognize incorrect predictions
- Be able to use dimensional analysis to simplify finite element simulations

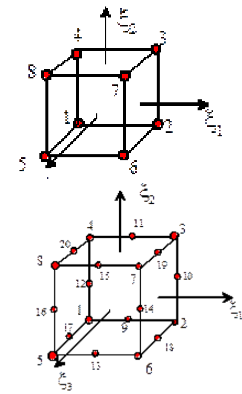
FEA Analysis

Features of FE Mesh

Nodes: Used to track motion of points in solid
Elements: Main purpose is to interpolate displacements between values at nodes.

ABAQUS offers linear (nodes at corners) and quadratic (nodes at mid-sides) elements



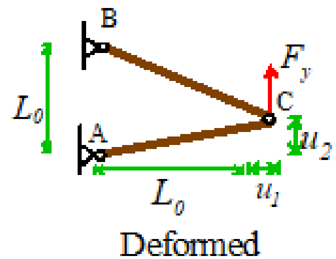


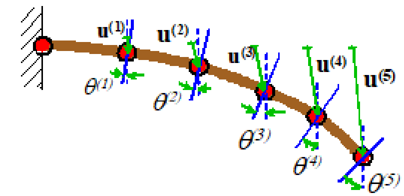
Linear

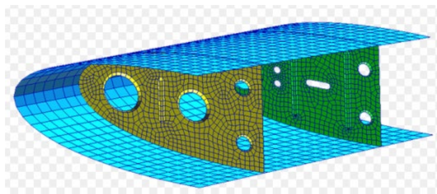
Quadratic

Special element types

Truss: Special displacement interpolation for 2 force members
Beam: Special displacement interpolation for slender member. Have rotation DOFS/moments
Plate/Shell: Special displacement interpolation for thin sheets that can deform out-of-plane. Rotations/moments

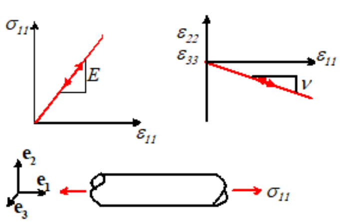




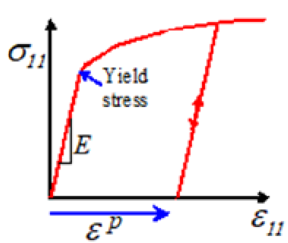


Materials (Some examples)

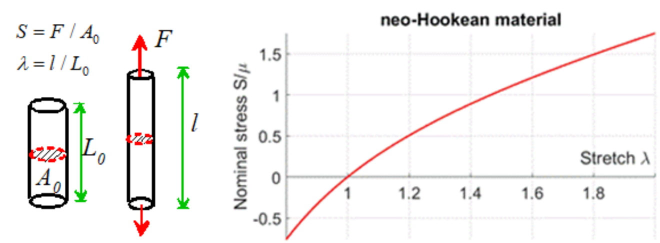
Linear Elasticity: OK for most materials subjected to small loads
Plasticity: Metals beyond yield
Hyperelasticity: Large strain reversible model used for rubbers
Viscoelasticity: Time dependent material used for polymers/tissue



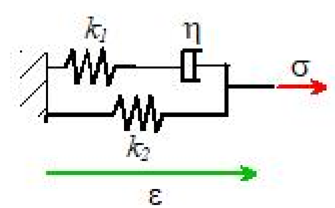
Linear Elastic



Elastic-Plastic



Hyperelastic

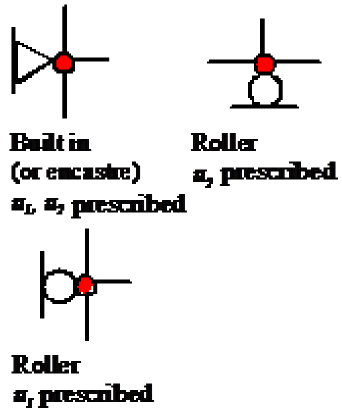


Visco-elastic

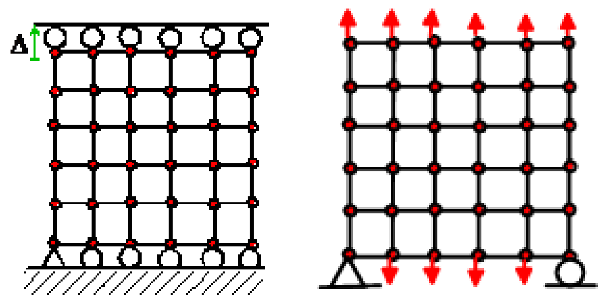
FEA Analysis

Boundary Conditions

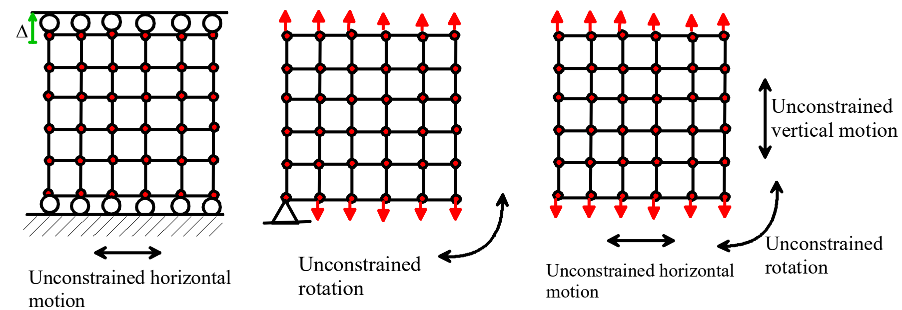
We can apply
1. Prescribed displacements
2. Forces on nodes
3. Pressure on element faces
4. Body forces
5. For some elements, can apply rotations/moments



Properly constrained solids



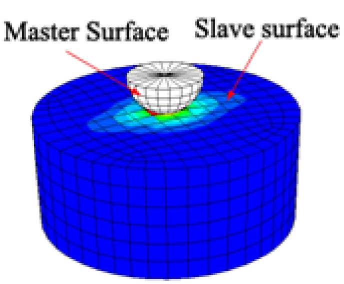
For static analysis we have to make sure we stop solid from translating/rotating



Incorrectly constrained solids

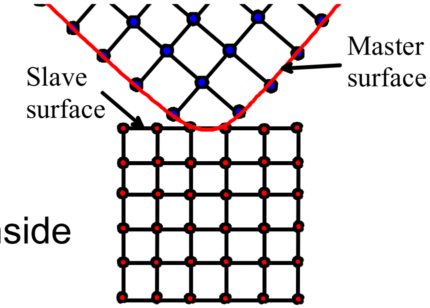
Contact

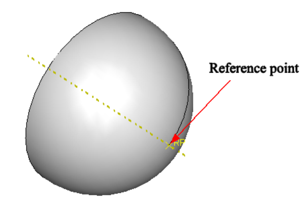
Select
1. Contact algorithm (Surface/Node Based)
2. Constitutive law for contact
   - "Soft" or "Hard" normal contact
   - Friction law



Master/slave pairs

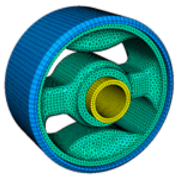
Nodes on slave surface are prevented from penetrating inside master surface

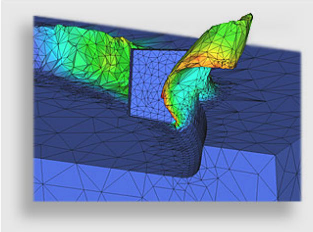




FEA Analysis

Solution Procedures





Small strain –v- large strain (NLGEOM)

Static analysis

Solves $\mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using Newton-Raphson iteration

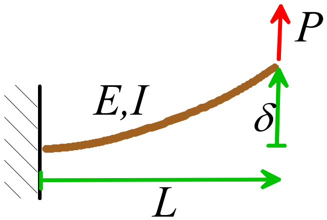
Explicit Dynamics: solves $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using 2nd order forward Euler scheme

Implicit Dynamics: solves $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{R}(\mathbf{u}) = \mathbf{F}^*$ $\quad \mathbf{u} = \mathbf{u}^*$ using 2nd order backward Euler scheme

Special procedures: modal dynamics, buckling (‘Linear Perturbation steps’)

Using Dimensional Analysis

Input data to code $\delta = f(P, E, I, L)$



Dimensionless form (others are possible) $\dfrac{\delta}{L} = f\left(\dfrac{P}{EL^2}, \dfrac{I}{L^4}\right)$

If we know $\delta = f(P, EI, L)$

Then $\dfrac{\delta}{L} = f\left(\dfrac{PL^2}{EI}\right)$

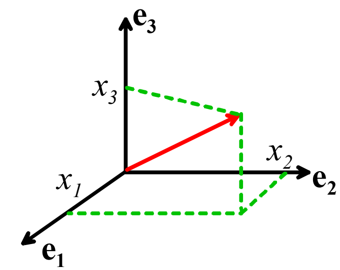
If we know problem is linear, then $\dfrac{\delta}{L} = C\dfrac{PL^2}{EI}$ for some constant C

Concept Checklist

3. Mathematics

- Understand the concepts of scalar, vector and tensor fields; understand use of Cartesian and polar basis vectors to represent vector and tensor fields
- Be able to compute gradient and divergence of scalar, and vector fields in Cartesian and Polar coordinates;
- Understand the concept of a tensor as a linear mapping of vectors;
- Be able to create a tensor using vector dyadic products; be able to add, subtract, multiply tensors; be able to calculate contracted products of tensors; be able to find the determinant, eigenvalues and eigenvectors of tensors; understand the spectral decomposition of a symmetric tensor;
- Be familiar with special tensors (identity, symmetric, skew, and orthogonal)
- Be able to transform tensor components from one basis to another.
- Be familiar with the conventions of index notation and perform simple algebra with index notation
- Be able to calculate the divergence of a symmetric tensor field in Cartesian or polar coordinates (eg to check the stress equilibrium equation)

Math



Position $\mathbf{r} = x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$

Scalar Field $\phi(x_i)$ gradient $\nabla \phi = \dfrac{\partial \phi}{\partial x_i} \mathbf{e}_i$

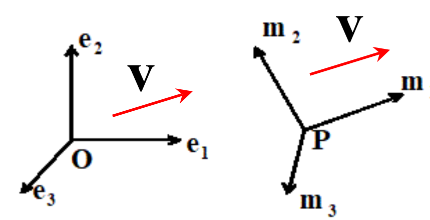
Vector Field $\mathbf{v}(x_i)$ gradient $\nabla \mathbf{v} = \dfrac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$

Tensor: linear map of vectors onto vectors $\mathbf{v} = \mathbf{S}\mathbf{u} \equiv v_i = S_{ij} u_j$ $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$

Dyadic product of vectors $\mathbf{S} = (\mathbf{a} \otimes \mathbf{b})$ $\mathbf{S}\mathbf{u} = (\mathbf{a} \otimes \mathbf{b})\mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\mathbf{a}$ $S_{ij} = a_i b_j$

General tensor as a sum of dyads $\mathbf{S} = S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$

Basis change formulas



Vectors: $\mathbf{v} = v_i^{(\mathbf{m})} \mathbf{m}_i = v_i^{(\mathbf{e})} \mathbf{m}_i$ $v_i^{(\mathbf{m})} = Q_{ij} v_j^{(\mathbf{e})}$

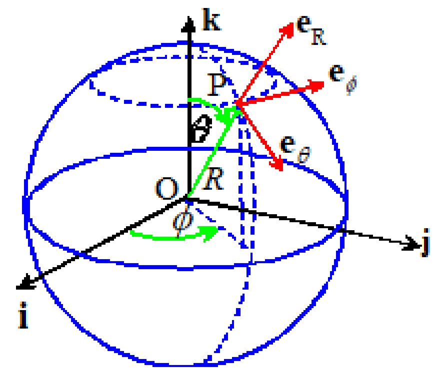
Tensors: $\mathbf{S} = S_{ij}^{(\mathbf{m})} \mathbf{m}_i \otimes \mathbf{m}_j = S_{ij}^{(\mathbf{e})} \mathbf{e}_i \otimes \mathbf{e}_j$

$S_{kl}^{(\mathbf{m})} = Q_{ki} S_{ij}^{(\mathbf{e})} Q_{lj}$ $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ $[Q] = \begin{bmatrix} \mathbf{m}_1 \cdot \mathbf{e}_1 & \mathbf{m}_1 \cdot \mathbf{e}_2 & \mathbf{m}_1 \cdot \mathbf{e}_3 \\ \mathbf{m}_2 \cdot \mathbf{e}_1 & \mathbf{m}_2 \cdot \mathbf{e}_2 & \mathbf{m}_2 \cdot \mathbf{e}_3 \\ \mathbf{m}_3 \cdot \mathbf{e}_1 & \mathbf{m}_3 \cdot \mathbf{e}_2 & \mathbf{m}_3 \cdot \mathbf{e}_3 \end{bmatrix}$

Gradients in Polar Coordinates

Position $\mathbf{r} = R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k}$

Vector $\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi$



Gradient of a scalar $\nabla f = \mathbf{e}_R\frac{\partial f}{\partial R} + \mathbf{e}_\theta\frac{1}{R}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{R\sin\theta}\frac{\partial f}{\partial \phi}$

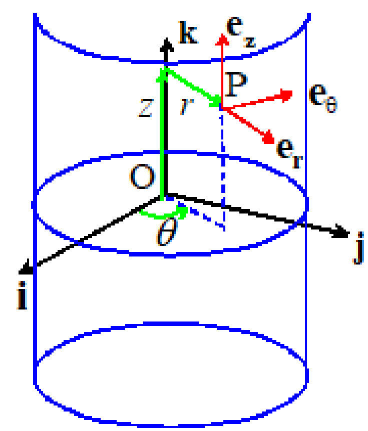
Gradient of a vector

$$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R} & \frac{1}{R\sin\theta}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R} \\ \frac{\partial v_\theta}{\partial R} & \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R} & \frac{1}{R\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \cot\theta\frac{v_\phi}{R} \\ \frac{\partial v_\phi}{\partial R} & \frac{1}{R}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta\frac{v_\theta}{R} + \frac{v_R}{R} \end{bmatrix}$$

Divergence of a vector $\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_R}{\partial R} + \frac{2v_R}{R} + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \cot\theta\frac{v_\theta}{R} + \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Position $\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$

Vector $\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z$



Gradient of a scalar $\nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z\frac{\partial f}{\partial z}$

Gradient of a vector

$$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector $\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$

Tensor Operations

Operations on 3x3 matrices also apply to tensors

Addition $\mathbf{U} = \mathbf{S} + \mathbf{T}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} S_{11}+T_{11} & S_{12}+T_{12} & S_{13}+T_{13} \\ S_{21}+T_{21} & S_{22}+T_{22} & S_{23}+T_{23} \\ S_{31}+T_{31} & S_{32}+T_{32} & S_{33}+T_{33} \end{bmatrix}$$

Vector/Tensor product $\mathbf{v} = \mathbf{S}\mathbf{u}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} S_{11}u_1 + S_{12}u_2 + S_{13}u_3 \\ S_{21}u_1 + S_{22}u_2 + S_{23}u_3 \\ S_{31}u_1 + S_{32}u_2 + S_{33}u_3 \end{bmatrix}$$

$\mathbf{v} = \mathbf{u}\mathbf{S}$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} u_1 S_{11} + u_2 S_{21} + u_3 S_{31} \\ u_1 S_{12} + u_2 S_{22} + u_3 S_{32} \\ u_1 S_{13} + u_2 S_{23} + u_3 S_{33} \end{bmatrix}$$

Tensor product $\mathbf{U} = \mathbf{T}\mathbf{S}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} & T_{11}S_{12} + T_{12}S_{22} + T_{13}S_{32} & T_{11}S_{13} + T_{12}S_{23} + T_{13}S_{33} \\ T_{21}S_{11} + T_{22}S_{21} + T_{23}S_{31} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} \\ T_{31}S_{11} + T_{32}S_{21} + T_{33}S_{31} & T_{31}S_{12} + T_{32}S_{22} + T_{33}S_{32} & T_{31}S_{13} + T_{32}S_{23} + T_{33}S_{33} \end{bmatrix}$$

Tensor Operations

Transpose
$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^T = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u}$$

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Determinant
$$\det(\mathbf{S}) = S_{11}(S_{22}S_{33} - S_{23}S_{32}) - S_{22}(S_{12}S_{33} - S_{32}S_{13}) + S_{33}(S_{12}S_{23} - S_{22}S_{13})$$

Eigenvalues/vectors
$$\mathbf{Sm} = \lambda \mathbf{m}$$

$$\det(\mathbf{S} - \lambda \mathbf{I}) = 0$$

Spectral decomposition for symmetric **S**
$$\mathbf{S} = \sum_{i=1}^{3} \lambda_i \mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)}$$

Inverse
$$\mathbf{S}^{-1} \cdot \mathbf{S} = \mathbf{I}$$

Identity
$$\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric tensor
$$\mathbf{S} = \mathbf{S}^T$$

Skew tensors
$$\mathbf{S}^T = -\mathbf{S}$$

Proper orthogonal tensors
$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

$$\det(\mathbf{R}) = +1$$

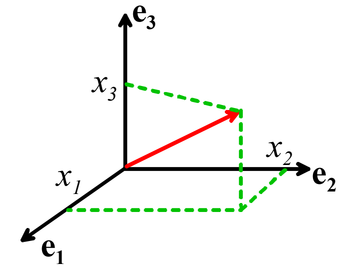
Inner product
$$\mathbf{S} : \mathbf{S} \equiv S_{ij}S_{ij} = S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + \ldots.$$

Outer product
$$\mathbf{S} \cdot\cdot \mathbf{S} \equiv S_{ij}S_{ji} = S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + \ldots.$$

Index Notation Summary

Vector $\mathbf{x} = (x_1, x_2, x_3)$

Tensor $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$



Index Notation $\mathbf{x} \equiv x_i$ $\quad \mathbf{S} \equiv S_{ij}$

Summation convention

$$\lambda = a_i b_i \equiv \lambda = \sum_{i=1}^{3} a_i b_i \equiv \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$$

$$c_i = S_{ik} x_k \equiv c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \begin{cases} c_1 = S_{11} x_1 + S_{12} x_2 + S_{13} x_3 \\ c_2 = S_{21} x_1 + S_{22} x_2 + S_{23} x_3 = \mathbf{S}\mathbf{x} \\ c_3 = S_{31} x_1 + S_{32} x_2 + S_{33} x_3 \end{cases}$$

$$\lambda = S_{ij} S_{ij} \equiv \lambda = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \equiv \lambda = S_{11} S_{11} + S_{12} S_{12} + \ldots + S_{31} S_{31} + S_{32} S_{32} + S_{33} S_{33} = \mathbf{S} : \mathbf{S}$$

$$C_{ij} = A_{ik} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \equiv \mathbf{C} = \mathbf{A}\mathbf{B} \qquad C_{ij} = A_{ki} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \equiv \mathbf{C} = \mathbf{A}\mathbf{B}^T$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad a_i \delta_{ij} = a_j \qquad \frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

Permutation symbol

$$\in_{ijk} = \begin{cases} 1 & i,j,k = 1,2,3; \quad 2,3,1 \quad \text{or} \quad 3,1,2 \\ -1 & i,j,k = 3,2,1; \quad 2,1,3 \quad \text{or} \quad 1,3,2 \\ 0 & \text{otherwise} \end{cases}$$

$$\in_{ijk} = \in_{kij} = \in_{jki} = -\in_{jik} = -\in_{kji} = -\in_{kji}$$

$$\in_{kki} = 0$$

$$\in_{ijk} \in_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{mk}$$

$$\lambda = \det \mathbf{A} \equiv \lambda = \frac{1}{6} \in_{ijk} \in_{lmn} A_{li} A_{mj} A_{nk} \qquad S_{ij}^{-1} = \frac{1}{2 \det(\mathbf{S})} \in_{ipq} \in_{jkl} S_{pk} S_{ql}$$

Concept Checklist

4. Deformations

- Understand the concept and definition of a deformation gradient; be able to calculate a deformation gradient from a displacement field in Cartesian/polar coordinates; be able to calculate understand the physical significance of the Jacobian of the deformation gradient
- Understand Lagrange strain and its physical significance; be able to calculate Lagrange strain from deformation gradient tensor or displacement measurements.
- Know the definition of the infinitesimal strain tensor; understand that it is an approximate measure of deformation; be able to calculate infinitesimal strains from a displacement in Cartesian/polar coords
- Know and understand the significance of the compatibility equation for infinitesimal strain in 2D, and be able to integrate 2D infinitesimal strain fields to calculate a displacement field.
- Be able to calculate principal strains and understand their physical significance
- Be able to transform strain components from one basis to another

Deformations

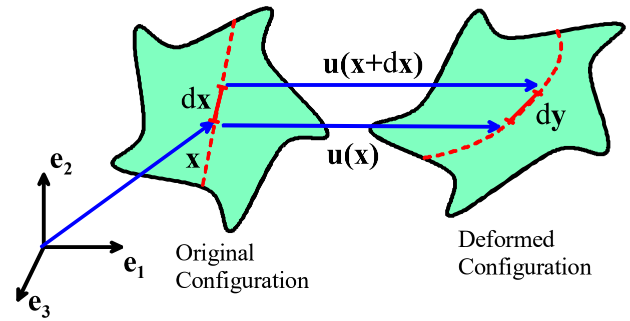
Deformation Mapping: $\mathbf{y}(\mathbf{x},t)$

Displacement Vector: $\mathbf{u}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - \mathbf{x}$

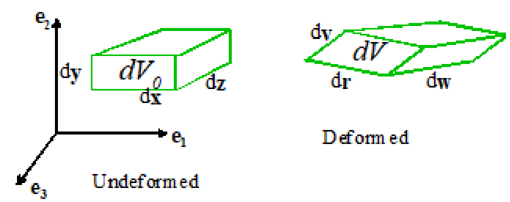
Deformation Gradient: $\mathbf{F} = \nabla\mathbf{y} = \nabla\mathbf{u} + \mathbf{I}$

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} + \delta_{ij}$$

$$d\mathbf{y} = \mathbf{F}d\mathbf{x} \qquad dy_i = F_{ij}dx_j$$



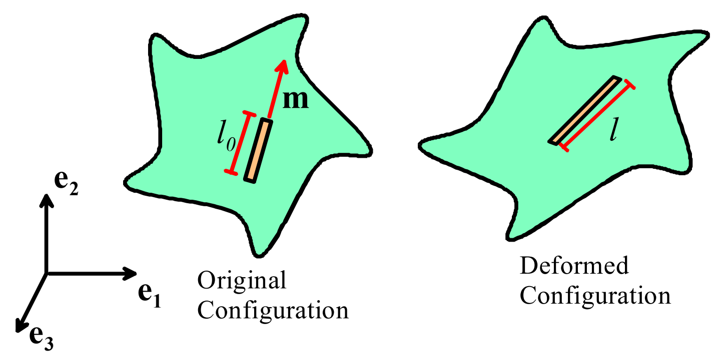
Jacobian: $J = \det(\mathbf{F}) \quad dV = JdV_0$



Lagrange Strain: $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$$

$$\frac{l^2 - l_0^2}{2l_0^2} = \mathbf{m}\cdot\mathbf{E}\mathbf{m} = m_i E_{ij} m_j$$

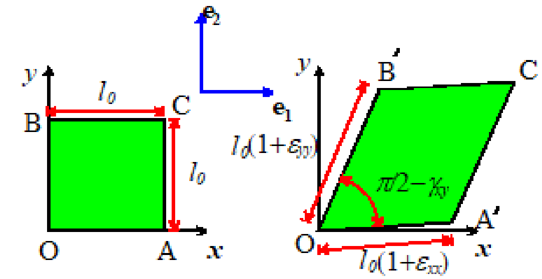


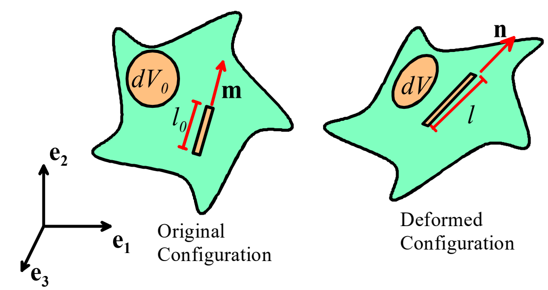
Can use this to find **E** given $l, l_0, \mathbf{m}$ for 3 (in 2D) directions

Deformations

Infinitesimal strain: $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Properties:
(1) Approximate strain measure used only for small deformation
(2) For small strains $\boldsymbol{\varepsilon} \approx \mathbf{E}$
(3) Components quantify length and angle changes of unit cube
(4) $\mathbf{m} \cdot \boldsymbol{\varepsilon}\mathbf{m} = m_i \varepsilon_{ij} m_j \approx (l - l_0)/l_0$
(5) $trace(\boldsymbol{\varepsilon}) = \varepsilon_{kk} \approx (dV - dV_0)/dV_0$





2D Compatibility conditions

To be able to integrate strains (to find displacement)

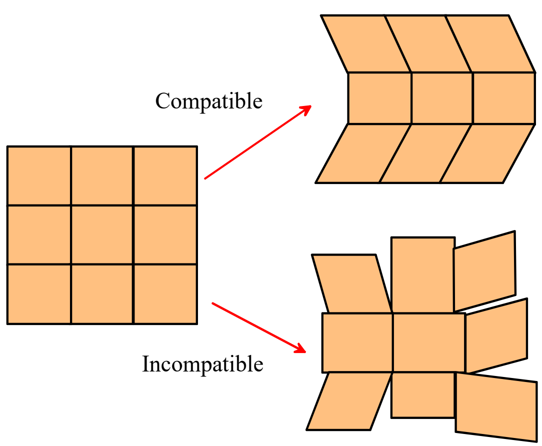
$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

Integrating strains

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \Rightarrow u_1 = \int \varepsilon_{11} dx_1 + f(x_2)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \Rightarrow u_2 = \int \varepsilon_{22} dx_2 + g(x_1)$$

Find *f,g* using $2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}$

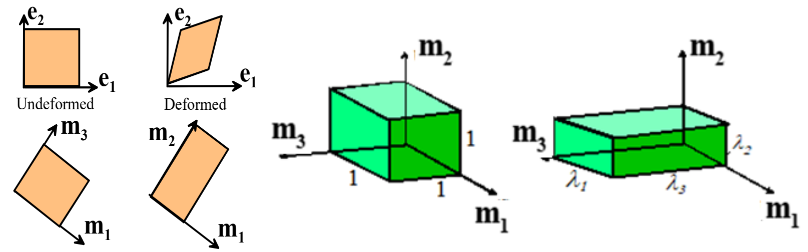


Deformations

Principal Strains and stretches

In principal basis $\{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)}\}$ strains are diagonal

$$\varepsilon_{ij}^{(\mathbf{m})} \equiv \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} \qquad E_{ij}^{(\mathbf{m})} \equiv \frac{1}{2}\begin{bmatrix} \lambda_1^2 - 1 & 0 & 0 \\ 0 & \lambda_2^2 - 1 & 0 \\ 0 & 0 & \lambda_3^2 - 1 \end{bmatrix}$$



Infinitesimal strain $\boldsymbol{\varepsilon}\mathbf{m}^{(i)} = e_i \mathbf{m}^{(i)}$

Lagrange strain $\mathbf{E}\mathbf{m}^{(i)} = \frac{1}{2}(\lambda_i^2 - 1)\mathbf{m}^{(i)}$

(eigenvalues – use usual method to find them)

Concept Checklist

5. Forces

- Understand the concepts of external surface traction and internal body force;
- Understand the concept of internal traction inside a solid.
- Understand how Newton's laws imply the existence of the Cauchy stress tensor
- Be able to calculate tractions acting on an internal plane with given orientation from the Cauchy stress tensor
- Be able to integrate tractions exerted by stresses over a surface to find the resultant force
- Know the definition of principal stresses, be able to calculate values of principal stress and their directions, understand the physical significance of principal stresses
- Know the definition of Hydrostatic stress and von Mises stress
- Understand the use of stresses in simple failure criteria (yield and fracture)
- Understand the boundary conditions for stresses at an exterior surface

6. Equations of motion for solids

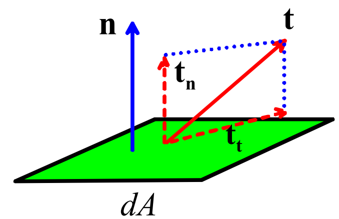
- Know the equations for linear momentum balance and angular momentum balance for a deformable solid
- Understand the significance of the small deformation approximation of the general equations of motion
- Know the equations of motion and static equilibrium for stress in Cartesian and polar coordinates
- Be able to check whether a stress field satisfies static equilibrium

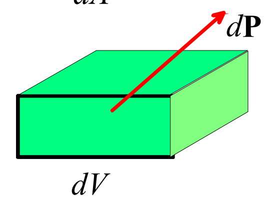
Describing external and internal forces

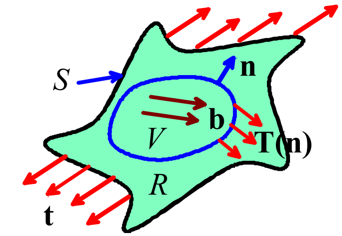
External Loading

Surface Traction $\mathbf{t} = \mathbf{t}_t + t_n \mathbf{n} = \lim_{dA \to 0} \frac{d\mathbf{P}}{dA}$

Body force (per unit mass) $\mathbf{b} = \lim_{dV \to 0} \frac{d\mathbf{P}}{\rho dV}$







Deformed Configuration

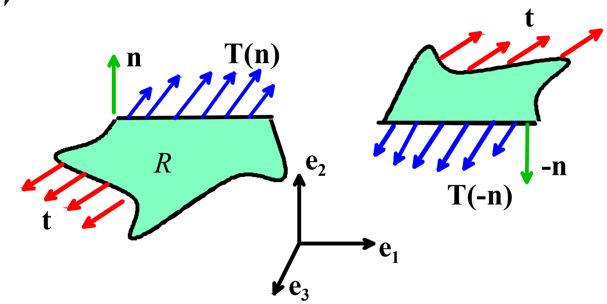
Internal Traction Vector T(n)

Quantifies force per unit area at a point on internal plane

Traction depends on direction of normal to surface

Satisfies: $\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n})$

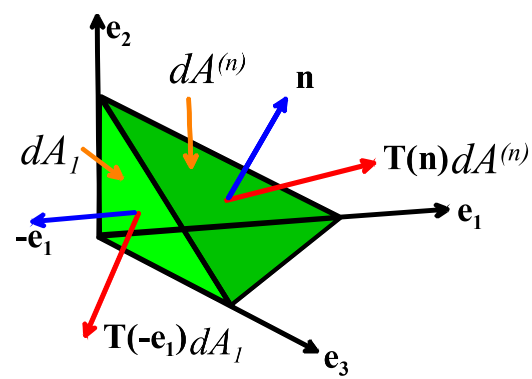
$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{e}_1) n_1 + \mathbf{T}(\mathbf{e}_2) n_2 + \mathbf{T}(\mathbf{e}_3) n_3$$



Cauchy ("True") Stress Tensor

Definition (components): $\sigma_{ij} = T_j(\mathbf{e}_i)$

Then: $T_j(\mathbf{n}) = n_i \sigma_{ij}$ $\qquad \mathbf{T} = \mathbf{n}\boldsymbol{\sigma}$

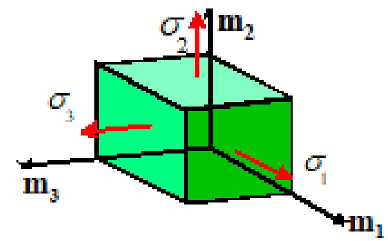


Warning: Some texts use transpose of this definition $\mathbf{T} = \boldsymbol{\sigma}\mathbf{n}$

Cauchy stress (force per unit deformed area) is symmetric $\sigma_{ij} = \sigma_{ji}$, so both are the same, but some other stresses eg nominal stress (force per unit undeformed area) are not, so be careful.

Stresses

Principal stresses (eigenvalues of stress tensor)



In principal basis stress is diagonal $\sigma_{ij}^{(\mathbf{m})} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$$\mathbf{n}^{(i)} \cdot \boldsymbol{\sigma} = \sigma_i \mathbf{n}^{(i)} \quad \text{or} \quad n_j^{(i)} \sigma_{jk} = \sigma_i n_k^{(i)} \quad (\text{no sum on } i)$$

If $\sigma_1 > \sigma_2 > \sigma_3$ then σ_1 is the largest stress acting normal to any plane

Hydrostatic stress $\sigma_h = \text{trace}(\boldsymbol{\sigma})/3 \equiv \sigma_{kk}/3$ $\quad \sigma_h = (\sigma_1 + \sigma_2 + \sigma_3)/3$

Deviatoric stress $\sigma'_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$

Von Mises stress $\sigma_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$ $\quad \sigma_e = \sqrt{\frac{1}{2}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right\}}$

Failure criterion for brittle materials (approximate) $\sigma_1 < \sigma_{frac}$

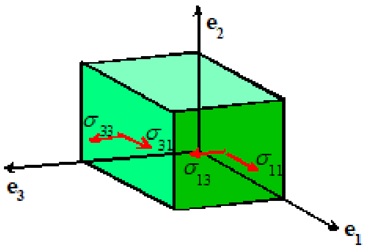
Yield criterion for metals (Von Mises) $\sigma_e < Y$

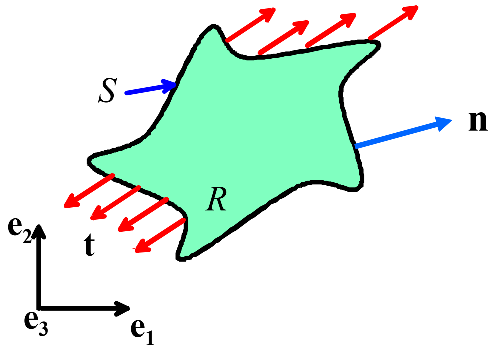
Stresses

Stresses near a boundary $n_i \sigma_{ij} = t_j$ $\quad \mathbf{n\sigma} = \mathbf{t}$

eg for $\mathbf{n} = \mathbf{e}_2 \quad \mathbf{t} = \mathbf{0}$

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = 0$$





Equations of motion

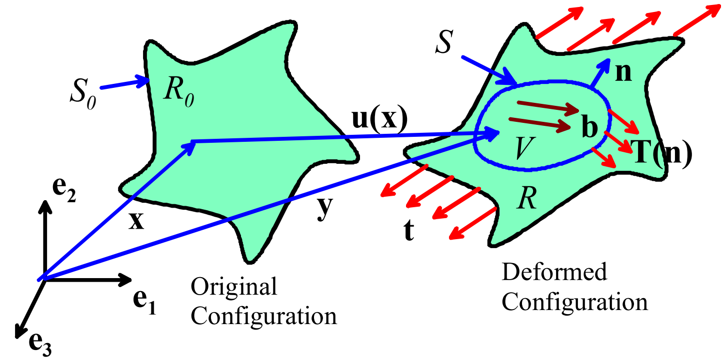
Linear Momentum

$$\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_i = \rho a_i$$

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho \frac{dv_1}{dt}$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho \frac{dv_2}{dt}$$

$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho \frac{dv_3}{dt}$$

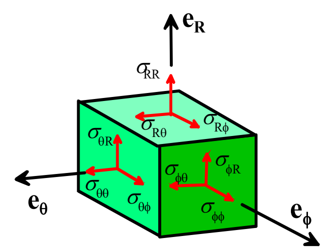


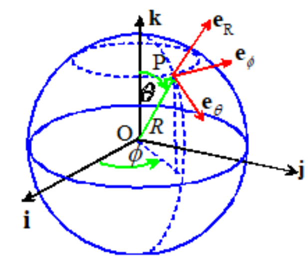
Angular Momentum $\sigma_{ij} = \sigma_{ji}$

Small deformations: replace **y** by **x** (approximate, but much easier to solve)

Spherical-polar coordinates

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{RR} & \sigma_{R\theta} & \sigma_{R\phi} \\ \sigma_{\theta R} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi R} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$



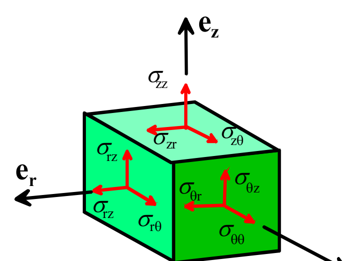


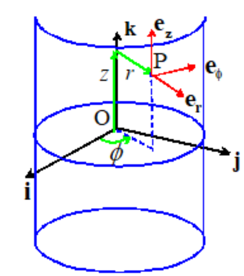
$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \equiv \begin{bmatrix} \frac{\partial \sigma_{RR}}{\partial R} + 2\frac{\sigma_{RR}}{R} + \frac{1}{R}\frac{\partial \sigma_{\theta R}}{\partial \theta} + \cot\theta \frac{\sigma_{\theta R}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi R}}{\partial \phi} - \frac{1}{R}(\sigma_{\theta\theta} + \sigma_{\phi\phi}) \\ \frac{\partial \sigma_{R\theta}}{\partial R} + 2\frac{\sigma_{R\theta}}{R} + \frac{1}{R}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \cot\theta \frac{\sigma_{\theta\theta}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta R}}{R} - \cot\theta\frac{\sigma_{\phi\phi}}{R} \\ \frac{\partial \sigma_{R\phi}}{\partial R} + 2\frac{\sigma_{R\phi}}{R} + \frac{\sin\theta}{R}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \cos\theta \frac{\sigma_{\theta\phi}}{R} + \frac{1}{R\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{R}(\sigma_{\phi R} + \sigma_{\phi\theta}) \end{bmatrix} + \begin{bmatrix} \rho b_R \\ \rho b_\theta \\ \rho b_\phi \end{bmatrix} = \begin{bmatrix} \rho \frac{dv_R}{dt} \\ \rho \frac{dv_\theta}{dt} \\ \rho \frac{dv_\phi}{dt} \end{bmatrix}$$

Cylindrical-polar coordinates

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \equiv \begin{bmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zR}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta r}}{r} + \frac{\partial \sigma_{z\theta}}{\partial z} \\ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} \end{bmatrix} + \begin{bmatrix} \rho b_r \\ \rho b_\theta \\ \rho bz \end{bmatrix} = \begin{bmatrix} \rho \frac{dv_r}{dt} \\ \rho \frac{dv_\theta}{dt} \\ \rho \frac{dv_z}{dt} \end{bmatrix}$$

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$





Concept Checklist

7. Stress-strain relations for elastic materials subjected to small strains

- Understand the concept of an isotropic material
- Understand the assumptions associated with idealizing a material as linear elastic
- Know the stress-strain-temperature equations for an isotropic, linear elastic solid
- Understand how to simplify stress-strain temperature equations for plane stress or plane strain deformation
- Be familiar with definitions of elastic constants (Young’s, shear, bulk and Lame moduli, Poissons ratio)
- Be able to calculate strain energy density of a stress or strain field in an elastic solid
- Be able to calculate stress/strain in an elastic solid subjected to uniform loading or temperature.

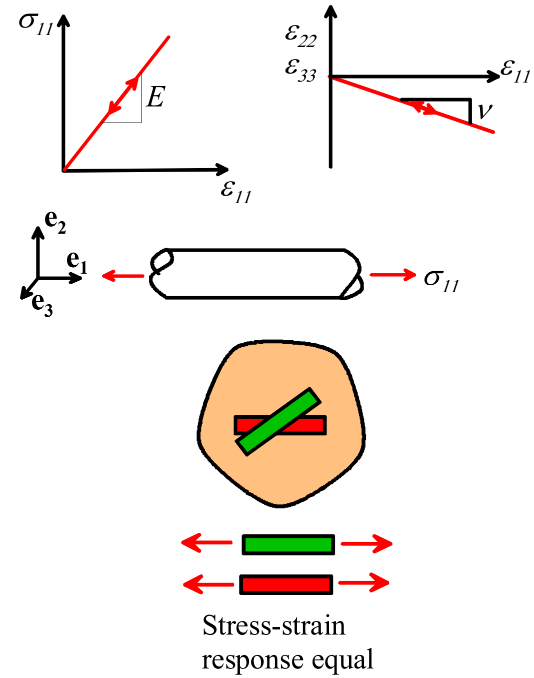
Stress-strain-temperature relations for elastic solids

Assumptions

Displacements/rotations are small – we can use infinitesimal strain as our deformation measure

Isotropy: Material response is independent of orientation of specimen with respect to underlying material

Elasticity: Material behavior is perfectly reversible, and relation between stress, strain and temperature is linear



Then $$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$

More generally $$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha\Delta T\delta_{kl}\right) \qquad \varepsilon_{ij} = S_{ijkl}\sigma_{kl} + \alpha\Delta T\delta_{ij}$$

Strain Energy Density

Separate strain into elastic and thermal parts $$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^T$$

$$\varepsilon_{ij}^e = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{ij}^T = \alpha\Delta T\delta_{ij}$$

Strain energy density $$U = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}^e$$

$$U = \frac{1+\nu}{2E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\sigma_{kk}\sigma_{jj}$$

$$U = \frac{E}{2(1+\nu)}\varepsilon_{ij}^e\varepsilon_{ij}^e + \frac{E\nu}{2(1+\nu)(1-2\nu)}\varepsilon_{jj}^e\varepsilon_{kk}^e$$

Useful elasticity formulas for isotropic materials

Matrix form for stress-strain law (3D)

$$\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\2\varepsilon_{23}\\2\varepsilon_{13}\\2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1&-\nu&-\nu&0&0&0\\-\nu&1&-\nu&0&0&0\\-\nu&-\nu&1&0&0&0\\0&0&0&2(1+\nu)&0&0\\0&0&0&0&2(1+\nu)&0\\0&0&0&0&0&2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{23}\\\sigma_{13}\\\sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu&\nu&\nu&0&0&0\\\nu&1-\nu&\nu&0&0&0\\\nu&\nu&1-\nu&0&0&0\\0&0&0&\frac{(1-2\nu)}{2}&0&0\\0&0&0&0&\frac{(1-2\nu)}{2}&0\\0&0&0&0&0&\frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\2\varepsilon_{23}\\2\varepsilon_{13}\\2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

For plane strain $\varepsilon_{33}=\varepsilon_{23}=\varepsilon_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\2\varepsilon_{12}\end{bmatrix}=\frac{(1+\nu)}{E}\begin{bmatrix}1-\nu&-\nu&0\\-\nu&1-\nu&0\\0&0&2\end{bmatrix}\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{bmatrix}+(1+\nu)\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu&\nu&0\\\nu&1-\nu&0\\0&0&\frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\sigma_{33}=\frac{E\nu(\varepsilon_{11}+\varepsilon_{22})}{(1-2\nu)(1+\nu)}+\frac{E\alpha\Delta T}{1-2\nu},\ \sigma_{13}=\sigma_{23}=0$$

For plane stress $\sigma_{33}=\sigma_{23}=\sigma_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1&-\nu&0\\-\nu&1&0\\0&0&2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{12}\end{bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&(1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{(1-\nu)}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\varepsilon_{33}=-\frac{\nu}{E}(\sigma_{11}+\sigma_{22})+\alpha\Delta T$$

Strain Energy Density

$$U=\frac{1}{2}\left[\sigma_{11}\varepsilon_{11}^e+\sigma_{22}\varepsilon_{22}^e+\sigma_{33}\varepsilon_{33}^e+2\sigma_{12}\varepsilon_{12}+2\sigma_{13}\varepsilon_{13}+2\sigma_{23}\varepsilon_{23}\right]$$

$$\varepsilon_{11}^e=\varepsilon_{11}-\alpha\Delta T\quad \varepsilon_{22}^e=\varepsilon_{22}-\alpha\Delta T\quad \varepsilon_{33}^e=\varepsilon_{33}-\alpha\Delta T$$

Relations between elastic constants

	LAME MODULUS λ	SHEAR MODULUS μ	YOUNG'S MODULUS E	POISSON'S RATIO ν	BULK MODULUS K
λ, μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ, E		Irrational		Irrational	Irrational
λ, ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ, K		$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	
μ, E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ, K	$\frac{3K-2\mu}{3}$		$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	
E, ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
E, K	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν, K	$\frac{3K\nu}{(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

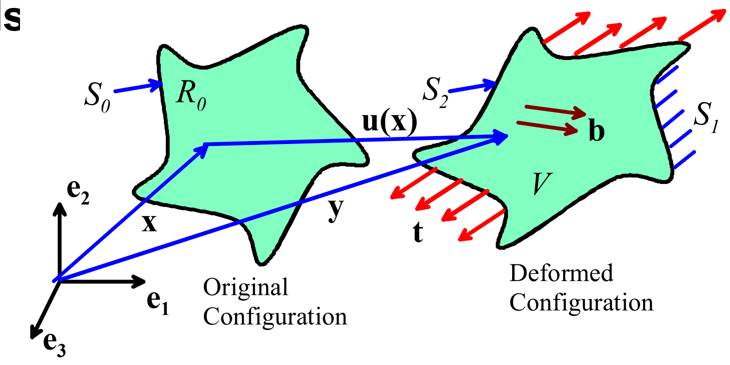
Concept Checklist

8. Analytical solutions to static problems for linear elastic solids

- Know the general equations (strain-displacement/compatibility, stress-strain relations, equilibrium) and boundary conditions that are used to calculate solutions for elastic solids
- Understand general features of solutions to elasticity problems: (1) solutions are linear; (2) solutions can be superposed; (3) Saint-Venants principle
- Know how to simplify the equations for spherically symmetric solids (using polar coords)
- Be able to calculate stress/strain in spherically or cylindrically symmetric solids under spherical/cylindrical symmetric loading by hand
- Understand how the Airy function satisfies the equations of equilibrium and compatibility for an elastic solid
- Be able to check whether an Airy function is valid, and be able to calculate stress/strain/displacements from an Airy function and check that the solution satisfies boundary conditions

Solutions for elastic solids

Static boundary value problems for linear elastic solids



Assumptions:

1. Small displacements
2. Isotropic, linear elastic material

Given:

1. Traction or displacement on all exterior surfaces
2. Body force and temperature distribution

Find: $[u_i, \varepsilon_{ij}, \sigma_{ij}]$

Governing Equations:

1. Strain-displacement relation (you can use the compatibility equation instead)

$$\varepsilon_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)/2 \qquad \boldsymbol{\varepsilon} = \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right]/2$$

2. Stress-strain law

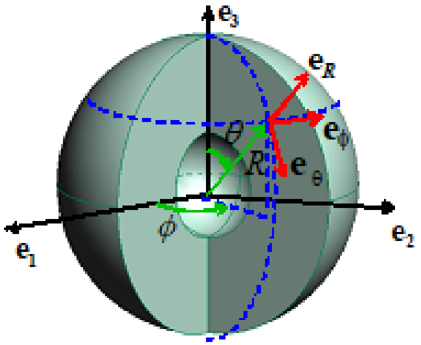
$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right) - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij} \qquad \boldsymbol{\sigma} = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}trace(\boldsymbol{\varepsilon})\mathbf{I}\right) - \frac{E\alpha\Delta T}{(1-2\nu)}\mathbf{I}$$

3. Equilibrium $\dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = 0 \qquad \nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0}$

4. Boundary conditions on external surfaces
    1. Where displacements are prescribed $u_i = u_i^* \qquad \mathbf{u} = \mathbf{u}^*$
    2. Where tractions are prescribed $n_j \sigma_{ji} = t_i \qquad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$

Solutions for elastic solids

Spherically symmetric solids



Position, displacement, body force

$$\mathbf{x} = R\mathbf{e}_R$$
$$\mathbf{u} = u(R)\mathbf{e}_R$$
$$\mathbf{b} = \rho_0 b(R)\mathbf{e}_R$$

Stress/strain

$$\sigma \equiv \begin{bmatrix} \sigma_{RR} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \varepsilon \equiv \begin{bmatrix} \varepsilon_{RR} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{\phi\phi} \end{bmatrix}$$

$$\varepsilon_{RR} = \frac{du}{dR} \qquad \varepsilon_{\phi\phi} = \varepsilon_{\theta\theta} = \frac{u}{R}$$

$$\sigma_{RR} = \frac{E}{(1+\nu)(1-2\nu)}\left\{(1-\nu)\varepsilon_{RR} + \nu\varepsilon_{\theta\theta} + \nu\varepsilon_{\phi\phi}\right\} - \frac{E\alpha\Delta T}{1-2\nu}$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{E}{(1+\nu)(1-2\nu)}\left\{\varepsilon_{\theta\theta} + \nu\varepsilon_{RR}\right\} - \frac{E\alpha\Delta T}{1-2\nu}$$

Equilibrium

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho_0 b_R = 0$$

$$\frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2u}{R^2} = \frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2 u\right)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{d\Delta T}{dR} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0 b(R)$$

Boundary conditions

$$u_R(a) = g_a \qquad u_R(b) = g_b$$

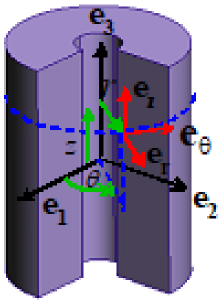
or $\sigma_{RR}(a) = t_a \qquad \sigma_{RR}(b) = t_b$

Solutions for elastic solids

Cylindrically symmetric solids

Position, displacement, body force

$$\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z$$
$$\mathbf{u} = u(r)\mathbf{e}_r + \varepsilon_{zz} z\mathbf{e}_z$$
$$\mathbf{b} = \rho_0 b(r)\mathbf{e}_r$$



Plane strain, or generalized plane strain

Stress/strain

$$\sigma \equiv \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \qquad \varepsilon \equiv \begin{bmatrix} \varepsilon_{rr} & 0 & 0 \\ 0 & \varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix} \qquad \varepsilon_{rr} = \frac{du}{dr} \qquad \varepsilon_{\theta\theta} = \frac{u}{r}$$

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ Plane strain

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} - \frac{E\alpha\Delta T}{1-\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ Plane stress

Equilibrium

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho_0 b_r = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{\partial \Delta T}{\partial r} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0(b+\omega^2 r)$$ Plane strain

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru)\right\} = \alpha(1+\nu)\frac{\partial \Delta T}{\partial r} - \frac{(1-\nu^2)}{E}\rho_0(b+\omega^2 r)$$ Plane stress

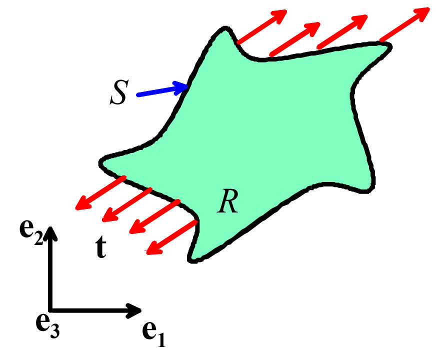
Boundary conditions

$$u_r(a) = g_a \qquad u_r(b) = g_b$$

or $\sigma_{rr}(a) = t_a \qquad \sigma_{rr}(b) = t_b$

Airy Function solution to elasticity problems

Airy Function $$\nabla^4\phi \equiv \frac{\partial^4\phi}{\partial x_1^4} + 2\frac{\partial^4\phi}{\partial x_1^2\partial x_2^2} + \frac{\partial^4\phi}{\partial x_2^4} = 0$$



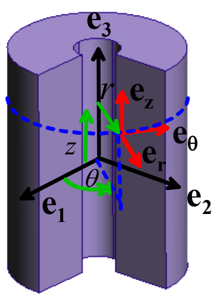
Stress $$\sigma_{11} = \frac{\partial^2\phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2\phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2\phi}{\partial x_1\partial x_2}$$

$$\sigma_{33} = 0 \qquad \text{(Plane Stress)}$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \qquad \text{(Plane Strain)}$$

$$\sigma_{23} = \sigma_{13} = 0$$

Airy Function $$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\phi = 0$$



Stress $$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

Strain $$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix} = \frac{(1+\nu)}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$$ Plane Strain

$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$$ Plane Stress

Displacement $$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \qquad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$