2.5 Units in FEA

FEA solves $\underline{F} = m\underline{a}$: No units a priori

Can use any convenient set : SI always safe

Be careful with lengths : if you sketch in mm
that sets length unit

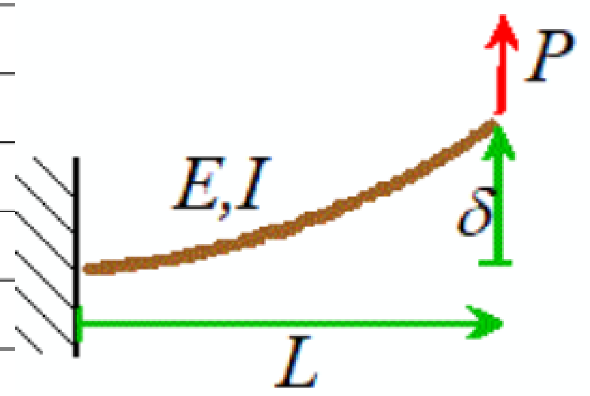
Example: Length : mm
Force : N
Stress : N/mm^2 : same as MPa
Density : $\dfrac{N}{\frac{mm}{s^2}\, mm^3} = \dfrac{Ns^2}{mm^4}$

eg $1000\ kg/m^3 = 1000 \times 10^{-12}\ \dfrac{Ns^2}{mm^4}$

Using dimensional analysis to simplify FEA

Consider simple example

Find deflection δ of elastic beam



We know $\delta = f(P, E, L, I)$

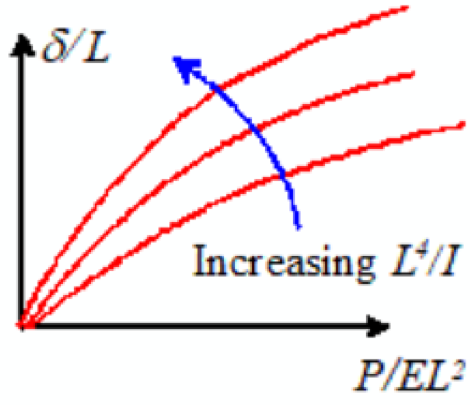
Goal: Reduce # variables by using dimensionless form

Units: $\delta = m$ $P : N$ $E : N/m^2$ $L : m$ $I : m^4$

$$\underbrace{\frac{\delta}{L}}_{m/m} = g\left(\underbrace{\frac{P}{E L^2}}_{\frac{N}{(N/m^2)\,m^2}} , \underbrace{\frac{L^4}{I}}_{\frac{m^4}{m^4}} \right)$$

⇐ 2 variables only

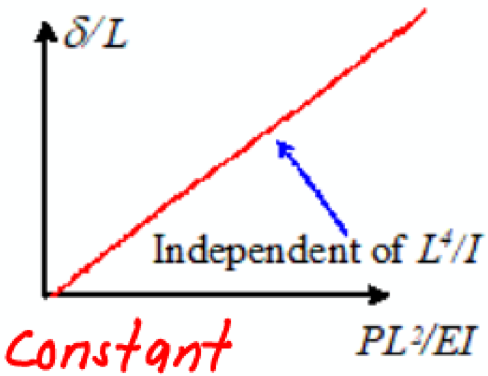
Results can be shown as a set of curves



If we know additional info we can simplify further

eg beam eq: $\underbrace{EI}_{\text{always appear together}} \dfrac{d^4w}{dx^4} = 0$

This means $\dfrac{\delta}{L} = f\left(\dfrac{PL^2}{EI}\right)$



Linear beam: δ proportional to P $\Rightarrow \dfrac{\delta}{L} = \beta \dfrac{PL^2}{EI}$

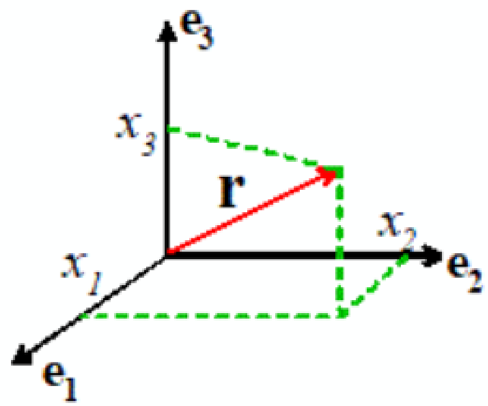
constant

Only need 1 FEA to find β

3 Math used in solid mechanics

Goals: Review some vector calculus & linear algebra
Introduce "Index Notation"
Introduce Tensors

3.1 Position Vector



Let $\{\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_3}\}$ be a Cartesian basis

$$\underset{\sim}{r} = x_1 \underset{\sim}{e_1} + x_2 \underset{\sim}{e_2} + x_3 \underset{\sim}{e_3}$$

Index notation $\underset{\sim}{r} = x_i \underset{\sim}{e_i}$

i can be 1, 2, 3

Repeated indices are summed

$$x_i \underset{\sim}{e_i} = \sum_{i=1}^{3} x_i \underset{\sim}{e_i}$$

3.2 Vector Operations

Dot Product $\underline{a} \cdot \underline{b} = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3$

Cross Product

Permutation Symbol $\epsilon_{ijk} = \begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ \text{all others zero} \end{cases}$

$$\underline{c} = \underline{a} \times \underline{b} \equiv c_i = \epsilon_{ijk} a_j b_k$$

Interpret : $c_1 = \epsilon_{111} a_1 b_1 + \epsilon_{112} a_1 b_2 + \cdots$ etc

$$= \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2$$

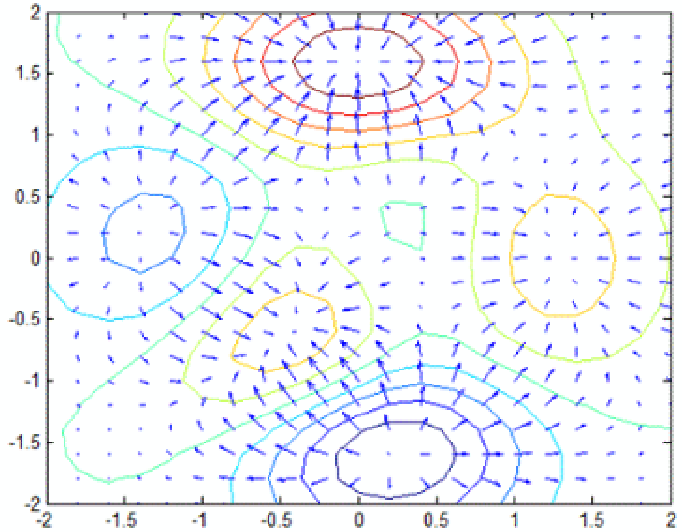
$$= a_2 b_3 - a_3 b_2$$

(same idea for c_2, c_3)

3.3 Vector Fields

Vector valued function of position

$$\underline{V}(\underline{r}) \quad \text{or} \quad V_i(x_j)$$



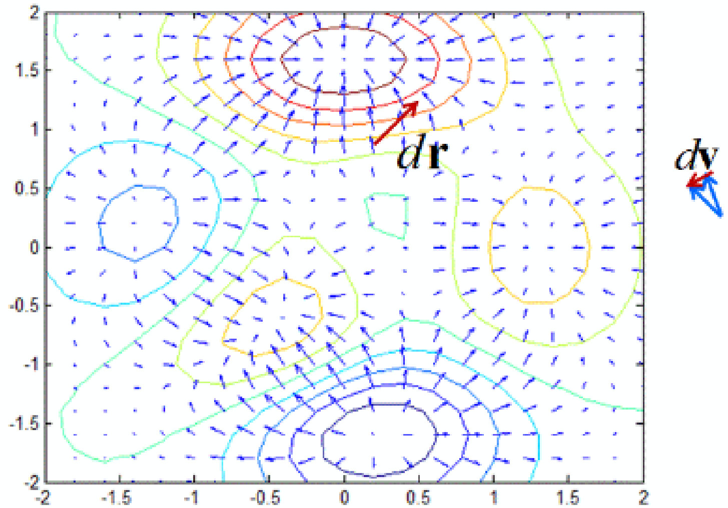
3.4 Gradient of a vector field

By definition gradient $G = \nabla \underline{V}$ has property that

$$d\underline{V} = \underline{G}\, d\underline{r}$$

for all infinitesimal $d\underline{r}$, $d\underline{V}$

$$d\underline{V} = (\nabla \underline{V})\, d\underline{r}$$



We can work out formulas for G using usual calculus rules

For example $dV_1 = \frac{\partial V_1}{\partial x_1} dx_1 + \frac{\partial V_1}{\partial x_2} dx_2 + \frac{\partial V_1}{\partial x_3} dx_3$

$$\Rightarrow \begin{bmatrix} dV_1 \\ dV_2 \\ dV_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial V_1/\partial x_1 & \partial V_1/\partial x_2 & \partial V_1/\partial x_3 \\ \partial V_2/\partial x_1 & \partial V_2/\partial x_2 & \\ \partial V_3/\partial x_1 & \text{etc} & \end{bmatrix}}_{\substack{\text{Components of } \nabla \underline{V} \\ 3\times 3 \text{ matrix}}} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

Index Notation $dV_i = G_{ij}\, dx_j$ (Tensor-matrix product)

$$G_{ij} = \frac{\partial V_i}{\partial x_j}.$$

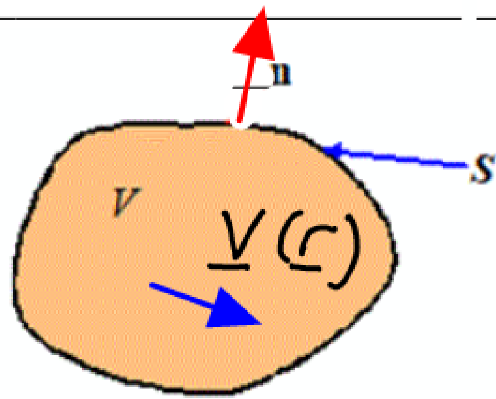
3.5 Divergence

$$\nabla \cdot \underline{V} = \text{trace}\,(\nabla \underline{V}) = \frac{\partial V_k}{\partial x_k} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}$$

If $\underline{V}$ is a flux $\nabla \cdot \underline{V}$ represents net flux into an infinitessimal vol element

3.6 Divergence Theorem

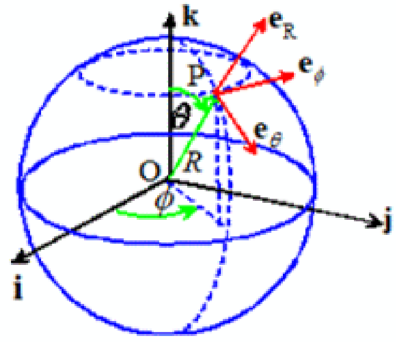
$$\int_{\bar{V}} (\nabla \cdot \underline{V})\, d\bar{V} = \int_S (\underline{n} \cdot \underline{V})\, dA$$



Gradients in Polar Coordinates

Position $\mathbf{r} = R\sin\theta\cos\phi\mathbf{i} + R\sin\theta\sin\phi\mathbf{j} + R\cos\theta\mathbf{k}$

Vector $\mathbf{a} = a_R\mathbf{e}_R + a_\theta\mathbf{e}_\theta + a_\phi\mathbf{e}_\phi$



Gradient of a scalar $\nabla f = \mathbf{e}_R\frac{\partial f}{\partial R} + \mathbf{e}_\theta\frac{1}{R}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi\frac{1}{R\sin\theta}\frac{\partial f}{\partial \phi}$

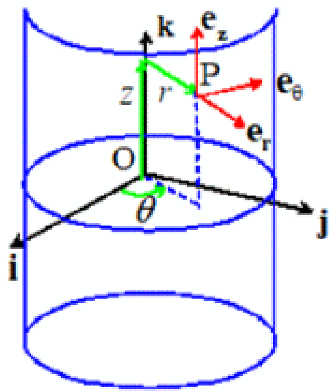
Gradient of a vector

$$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R} & \frac{1}{R\sin\theta}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R} \\ \frac{\partial v_\theta}{\partial R} & \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R} & \frac{1}{R\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \cot\theta\frac{v_\phi}{R} \\ \frac{\partial v_\phi}{\partial R} & \frac{1}{R}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta\frac{v_\theta}{R} + \frac{v_R}{R} \end{bmatrix}$$

Divergence of a vector $\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_R}{\partial R} + \frac{2v_R}{R} + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \cot\theta\frac{v_\theta}{R} + \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi}$

Position $\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + z\mathbf{k}$

Vector $\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta + a_z\mathbf{e}_z$



Gradient of a scalar $\nabla f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial \theta} + \mathbf{e}_z\frac{\partial f}{\partial z}$

Gradient of a vector

$$\nabla\mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Divergence of a vector $\nabla\cdot\mathbf{v} = trace(\nabla\mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$

3.7 Tensors

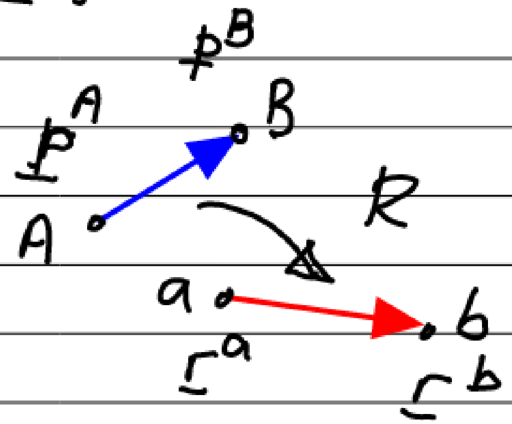
Definition: A tensor is a linear mapping of a vector $\underline{u}$ onto another vector $\underline{v}$

$$\underline{v} = S\,\underline{u} \qquad (\text{ or } \underline{v} = S \cdot \underline{u})$$

Examples: Gradient

Rotation tensor

$$(\underline{r}^b - \underline{r}^a) = R\,(\underline{p}^B - \underline{p}^A)$$



Tensor Components

Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ be a basis $\qquad \underline{u} = u_i \underline{e}_i$

$$\underline{v} = v_j \underline{e}_j$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}}_{\text{Components of } S} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Index notation $V_i = S_{ij} \, u_j$

Dyadic Product of vectors (tensor product)

Let $\underline{a}$, $\underline{b}$ be vectors

Define $S = \underline{a} \otimes \underline{b}$ such that

$$S\underline{u} = [\underline{a} \otimes \underline{b}]\,\underline{u} = \underline{a}\,(\underline{b} \cdot \underline{u})$$

Matrix form $S = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & \\ & & \text{etc} \end{bmatrix}$

Index Notation $S_{ij} = a_i b_j$

We can represent any tensor as a sum of 9 dyadic products of basis vectors

$$S = S_{ij} \, \underline{e}_i \otimes \underline{e}_j$$

Since components of S are a 3×3 matrix we can apply all matrix operations to S

Tensor Operations

Operations on 3x3 matrices also apply to tensors

Addition $\mathbf{U} = \mathbf{S} + \mathbf{T}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} S_{11}+T_{11} & S_{12}+T_{12} & S_{13}+T_{13} \\ S_{21}+T_{21} & S_{22}+T_{22} & S_{23}+T_{23} \\ S_{31}+T_{31} & S_{32}+T_{32} & S_{33}+T_{33} \end{bmatrix}$$

Vector/Tensor product $\mathbf{v} = \mathbf{S} \cdot \mathbf{u}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} S_{11}u_1 + S_{12}u_2 + S_{13}u_3 \\ S_{21}u_1 + S_{22}u_2 + S_{23}u_3 \\ S_{31}u_1 + S_{32}u_2 + S_{33}u_3 \end{bmatrix}$$

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{S} \quad \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \begin{bmatrix} u_1S_{11} + u_2S_{21} + u_3S_{31} \\ u_1S_{12} + u_2S_{22} + u_3S_{32} \\ u_1S_{13} + u_2S_{23} + u_3S_{33} \end{bmatrix}$$

Tensor product $\mathbf{U} = \mathbf{T} \cdot \mathbf{S}$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} & T_{11}S_{12} + T_{12}S_{22} + T_{13}S_{32} & T_{11}S_{13} + T_{12}S_{23} + T_{13}S_{33} \\ T_{21}S_{11} + T_{22}S_{21} + T_{23}S_{31} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} \\ T_{31}S_{11} + T_{32}S_{21} + T_{33}S_{31} & T_{31}S_{12} + T_{32}S_{22} + T_{33}S_{32} & T_{31}S_{13} + T_{32}S_{23} + T_{33}S_{33} \end{bmatrix}$$

Tensor Operations

Transpose

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^T = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u}$$

$$(\mathbf{A} \cdot \mathbf{B})^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

Determinant $\det(\mathbf{S}) = S_{11}(S_{22}S_{33} - S_{23}S_{32}) - S_{22}(S_{12}S_{33} - S_{32}S_{13}) + S_{33}(S_{12}S_{23} - S_{22}S_{13})$

Eigenvalues/vectors $\mathbf{S} \cdot \mathbf{m} = \lambda \mathbf{m}$

$$\det(\mathbf{S} - \lambda \mathbf{I}) = 0$$

Inverse $\mathbf{S}^{-1} \cdot \mathbf{S} = \mathbf{I}$

Identity $\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Symmetric tensor $\mathbf{S} = \mathbf{S}^T$ Skew tensors $\mathbf{S}^T = -\mathbf{S}$

Proper orthogonal tensors $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$ $\det(\mathbf{R}) = +1$

$$\mathbf{R}^{-1} = \mathbf{R}^T$$

Index Notation

Addition $\quad U_{ij} = S_{ij} + T_{ij}$

Multiplication $\quad U_{ij} = S_{ik} T_{kj} \qquad U = ST$

Transpose $\quad S_{ij} = T_{ji} \qquad S = T^T$

also $\quad U = S^T T \equiv U_{ij} = S_{ki} T_{kj}$

$U = S T^T \equiv U_{ij} = S_{ik} T_{jk}$