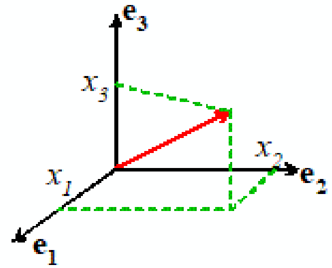
Review



Position $\mathbf{r} = x_i \mathbf{e}_i = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$

Scalar Field $\phi(x_i)$ gradient $\nabla \phi = \dfrac{\partial \phi}{\partial x_i} \mathbf{e}_i$

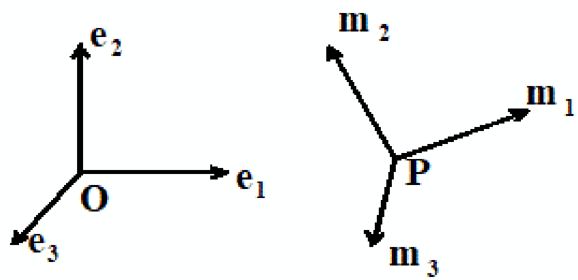
Vector Field $\mathbf{v}(x_i)$ gradient $\nabla \mathbf{v} = \dfrac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j$

Tensor: linear map of vectors onto vectors $\mathbf{v} = \mathbf{S} \cdot \mathbf{u} \equiv v_i = S_{ij} u_j$ $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$

Dyadic product of vectors $\mathbf{S} = (\mathbf{a} \otimes \mathbf{b})$ $\quad \mathbf{S} \cdot \mathbf{u} = (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{u} = (\mathbf{b} \cdot \mathbf{u})\mathbf{a}$ $\quad S_{ij} = a_i b_j$

General tensor as a sum of dyads $\mathbf{S} = S_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$

3.8 Basis Change Formulas



Tensor components depend on choice of basis

Need formulas to change basis

Vectors

$$\underline{v} = \underbrace{v_i^{(\underline{m})}}_{\text{Components in } \{\underline{m}_1, \underline{m}_2, \underline{m}_3\}} \underline{m}_i = \underbrace{v_i^{(\underline{e})}}_{\text{Components in } \{\underline{e}_1, \underline{e}_2, \underline{e}_3\}} \underline{e}_i$$

Dot both sides with $\underline{m}_j$:

$$\underline{m}_j \cdot \underline{v} = v_i^{(\underline{m})} \underbrace{(\underline{m}_j \cdot \underline{m}_i)}_{\delta_{ji}} = v_i^{(\underline{e})} \underbrace{\underline{m}_j \cdot \underline{e}_i}_{Q_{ji}}$$

(3x3 matrix)

Kronecker Delta $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

(Identity)

Property : $\delta_{ji} a_i = a_j$ $\delta_{ij}\delta_{jk} = \delta_{ik}$

Hence $\boxed{V_j^{(\underline{m})} = Q_{ji} V_i^{(\underline{e})}}$

Matrix form $\begin{bmatrix} V_1^{(m)} \\ V_2^{m} \\ V_3^{m} \end{bmatrix} = \begin{bmatrix} \underline{m}_1 \cdot \underline{e}_1 & \underline{m}_1 \cdot \underline{e}_2 & \underline{m}_1 \cdot \underline{e}_3 \\ \underline{m}_2 \cdot \underline{e}_1 & \underline{m}_2 \cdot \underline{e}_2 & \\ & & \text{etc} \end{bmatrix} \begin{bmatrix} V_1^{\underline{e}} \\ V_2^{\underline{e}} \\ V_3^{\underline{e}} \end{bmatrix}$

Observation: Q is orthogonal

$$Q Q^T = Q^T Q = I$$

Tensors $\quad \underset{\sim}{S} = S_{ij}^{(m)}\, \underset{\sim}{m}_i \otimes \underset{\sim}{m}_j = S_{ij}^{(e)}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

Dot products

$$\underset{\sim}{m}_k \cdot \underset{\sim}{S}\, \underset{\sim}{m}_\ell = S_{ij}^{(m)} \underbrace{(\underset{\sim}{m}_k \cdot \underset{\sim}{m}_i)}_{\delta_{ki}} \underbrace{(\underset{\sim}{m}_\ell \cdot \underset{\sim}{m}_j)}_{\delta_{\ell j}} = S_{ij}^{(e)} \underbrace{(\underset{\sim}{m}_k \cdot \underset{\sim}{e}_i)}_{Q_{ki}} \underbrace{(\underset{\sim}{e}_j \cdot \underset{\sim}{m}_\ell)}_{Q_{\ell j}}$$

$$\boxed{\begin{aligned} S_{k\ell}^{(m)} &= Q_{ki}\, S_{ij}\, Q_{\ell j} \\ [S^{(m)}] &= [Q][S^{(e)}][Q]^T \end{aligned}}$$

Example: $\mathbf{v} = (x_1 + 2x_2)\mathbf{e}_1 + (x_1 - 2x_2)\mathbf{e}_2$

(a) Find $\nabla \mathbf{v}$

(b) Let $\mathbf{m}_1 = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2} \quad \mathbf{m}_2 = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$

Find components of **v** in $\{\mathbf{m}_1, \mathbf{m}_2\}$

Find components of $\nabla \mathbf{v}$ in $\{\mathbf{m}_1, \mathbf{m}_2\}$

Use 2x2 matrices

(a) Formula $\nabla \underset{\sim}{v} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{bmatrix}$

Hence $\nabla \underline{v} = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$

(b) Use formula $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$V^{(\underline{m})} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ x_1 - 2x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2x_1 \\ 4x_2 \end{bmatrix}$$

$$\nabla \underline{v}^{(\underline{m})} = [Q] [\nabla \underline{v}^{\underline{e}}] [Q]^T$$

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 2 \\ 4 & -4 \end{bmatrix}$$

3.9 Principal Values and directions of symmetric tensors

Many tensors are symmetric : strain, stress, Inertia etc

For a symmetric tensor we can find a special basis in which components are a diagonal matrix

In this basis $[S^{(\underline{m})}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$

$\sigma_1, \sigma_2, \sigma_3$ are "principal Values"

$\underline{m}_1, \underline{m}_2, \underline{m}_3$ are "principal directions"

Finding principal values & directions

Note $S\,\underline{m}_1 = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix} = \sigma_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \sigma_1\,\underline{m}_1$ (similarly $S\,\underline{m}_2 = \sigma_2\,\underline{m}_2$ etc)

Hence $(\sigma_1 \;\; \sigma_2 \;\; \sigma_3)$ are eigenvalues of S

$(\underline{m}_1\;\; \underline{m}_2 \;\; \underline{m}_3)$ " eigenvectors

```
S = magic(3) + transpose(magic(3))   % A symmetric matrix
[Qtrans,D] = eig(S) %Find eigenvectors (columns of Qtrans) and eigenvalues (in D)
Q = transpose(Qtrans) % This is the Q that changes basis from (e) to (m)
check = Q*S*transpose(Q) % check should be components of S in the principal basis - i.e. D
Qtrans*D*transpose(Qtrans) % This should produce S
```

```
S = 3×3

    16     4    10
     4    10    16
    10    16     4

Qtrans = 3×3

   -0.2113   -0.7887   -0.5774
   -0.5774    0.5774   -0.5774
    0.7887    0.2113   -0.5774

D = 3×3

  -10.3923         0         0
         0   10.3923         0
         0         0   30.0000

Q = 3×3

   -0.2113   -0.5774    0.7887
   -0.7887    0.5774    0.2113
   -0.5774   -0.5774   -0.5774

check = 3×3

  -10.3923    0.0000   -0.0000
    0.0000   10.3923   -0.0000
   -0.0000   -0.0000   30.0000

ans = 3×3

   16.0000    4.0000   10.0000
    4.0000   10.0000   16.0000
   10.0000   16.0000    4.0000
```

3.10 Contracted Products of tensors

Inner Product : $S : T = S_{ij} T_{ij} = S_{11} T_{11} + S_{12} T_{12} + S_{13} T_{13} + \dots etc$

Outer Product : $S \cdot\cdot T = S_{ij} T_{ji} = S_{11} T_{11} + S_{12} T_{21} + S_{13} T_{31} + \dots etc$

3.11 Tensor "Invariants"

A tensor invariant remains constant under a change of basis

Examples: trace (S) (sum of diagonals)

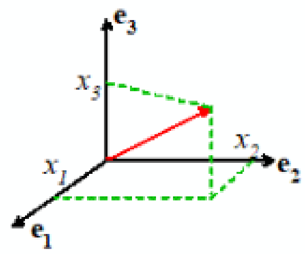
$S : S$

$\det(S)$

Eigenvalues

Index Notation Summary

Vector $\mathbf{x} = (x_1, x_2, x_3)$ Tensor $\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$



Index Notation $\mathbf{x} \equiv x_i$ $\mathbf{S} \equiv S_{ij}$

Summation convention

$$\lambda = a_i b_i \quad \equiv \quad \lambda = \sum_{i=1}^{3} a_i b_i \quad \equiv \quad \lambda = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$$

$$c_i = S_{ik} x_k \quad \equiv \quad c_i = \sum_{k=1}^{3} S_{ik} x_k \equiv \begin{cases} c_1 = S_{11}x_1 + S_{12}x_2 + S_{13}x_3 \\ c_2 = S_{21}x_1 + S_{22}x_2 + S_{23}x_3 = \mathbf{S} \cdot \mathbf{x} \\ c_3 = S_{31}x_1 + S_{32}x_2 + S_{33}x_3 \end{cases}$$

$$\lambda = S_{ij} S_{ij} \quad \equiv \quad \lambda = \sum_{i=1}^{3} \sum_{j=1}^{3} S_{ij} S_{ij} \quad \equiv \lambda = S_{11}S_{11} + S_{12}S_{12} + \ldots + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33} = \mathbf{S} : \mathbf{S}$$

$$C_{ij} = A_{ik} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj} \quad \equiv \quad \mathbf{C} = \mathbf{AB}$$

$$C_{ij} = A_{ki} B_{kj} \equiv C_{ij} = \sum_{k=1}^{3} A_{ki} B_{kj} \quad \equiv \quad \mathbf{C} = \mathbf{A}^T \mathbf{B}$$

Kronecker Delta $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $a_i \delta_{ij} = a_j$

Permutation symbol $\in_{ijk}$ $\begin{cases} \in_{123} = \in_{231} = \in_{312} = \ 1 \\ \in_{321} = \in_{213} = \in_{132} = -1 \\ \text{All others} \ = 0 \end{cases}$ $c_i = \in_{ijk} a_j b_k \equiv \mathbf{c} = \mathbf{a} \times \mathbf{b}$

Additional Properties of ϵ_{ijk}

$$\epsilon_{ijk}\,\epsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp}$$

$$\det(S) = \frac{1}{6}\,\epsilon_{ijk}\,\epsilon_{pqr}\,S_{ip}\,S_{jq}\,S_{kr}$$

$$S^{-1}_{ij} = \frac{1}{2\det(S)}\,\epsilon_{ik\ell}\,\epsilon_{jpq}\,S_{kp}\,S_{\ell q}$$

Example: Show that $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$

Use index formulas

$$\epsilon_{ijk}\,a_j\,b_k\;\epsilon_{ipq}\,c_p\,d_q = (\delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp})\,a_j\,b_k\,c_p\,d_q$$

$$= a_j c_j b_k d_k - a_j d_j b_k c_k$$

$$= (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$$

Example: Find $\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = \boxed{3}$

Example: Find $\dfrac{\partial x_i}{\partial x_j} = \boxed{\delta_{ij}}$

Example Let $r = \sqrt{x_i x_i}$ Find $\dfrac{\partial r}{\partial x_j}$

Chain rule $\dfrac{1}{2} \dfrac{1}{\sqrt{x_k x_k}} \dfrac{\partial (x_i x_i)}{\partial x_j} = \dfrac{1}{2} \dfrac{1}{r} 2x_i \underbrace{\dfrac{\partial x_i}{\partial x_j}}_{\delta_{ij}} = \dfrac{x_i}{r} \delta_{ij} = \boxed{\dfrac{x_j}{r}}$