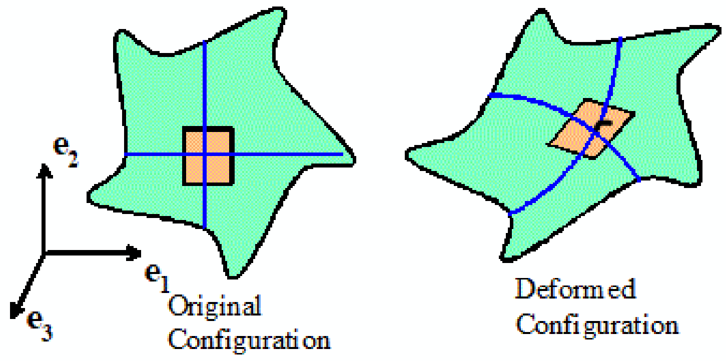
4 Mathematical Descriptions of Deformation

Background



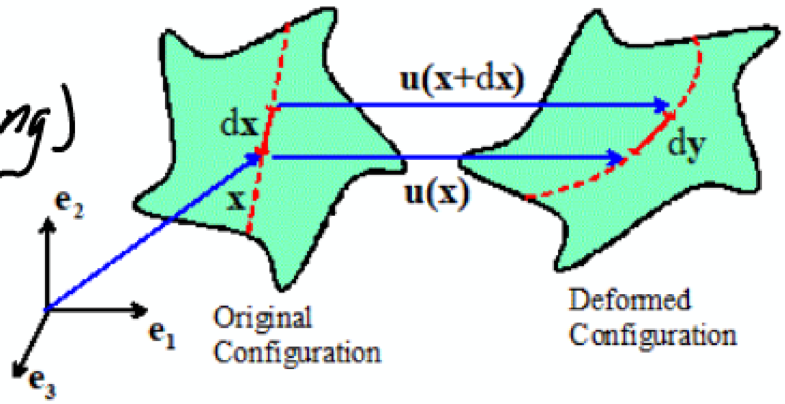
① "Local Action" stress at a point depends on deformation of infinitesimal Vol element

② Deformation is "Locally homogeneous" infinitesimal fibers remain straight; nearby parallel fibers remain parallel

⇒ Implication: We need to quantify deformation of infinitesimal vol elements

Large numbers of deformation measures exist - discuss a few

4.1 Deformation mapping and displacement field



Position before deformation $\underline{x}$

" after " $\underline{y}(\underline{x})$ (mapping)

Displacement $\underline{u}(\underline{x}) = \underline{y}(\underline{x}) - \underline{x}$

4.2 Deformation Gradient

Definition $F = \nabla y$ $\quad F_{ij} = \frac{\partial y_i}{\partial x_j}$ $\quad [F] = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \text{etc} & \end{bmatrix}$

Also $\underline{y}(\underline{x}) = \underline{x} + \underline{u}(\underline{x})$

$$F = I + \nabla \underline{u} \qquad F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}$$

Significance: F maps infinitessimal fiber $d\underline{x}$ onto deformed fiber dy

$$dy = F\,d\underline{x} \qquad dy_i = F_{ij}\,dx_j$$

To see this note

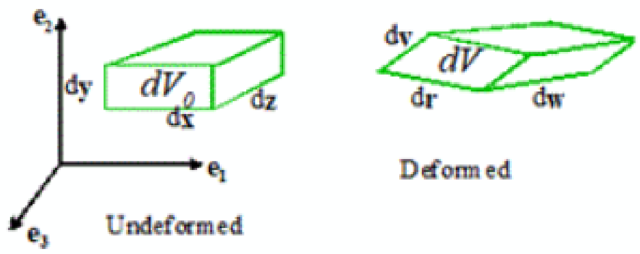
$$dy = y(\underline{x} + d\underline{x}) - y(\underline{x})$$

Taylor expansion $= y(\underline{x}) + \nabla y\, d\underline{x} + \cdots - y(\underline{x})$

$$dy \approx F\,d\underline{x}$$

F is used but not always the best choice for stress-strain laws

4.3 Jacobian of F J



Quantifies Vol changes

$$J = \det(F) = \frac{dV}{dV_0}$$

4.4 Lagrange Strain

Definition $\quad E = \frac{1}{2}(F^T F - I)$

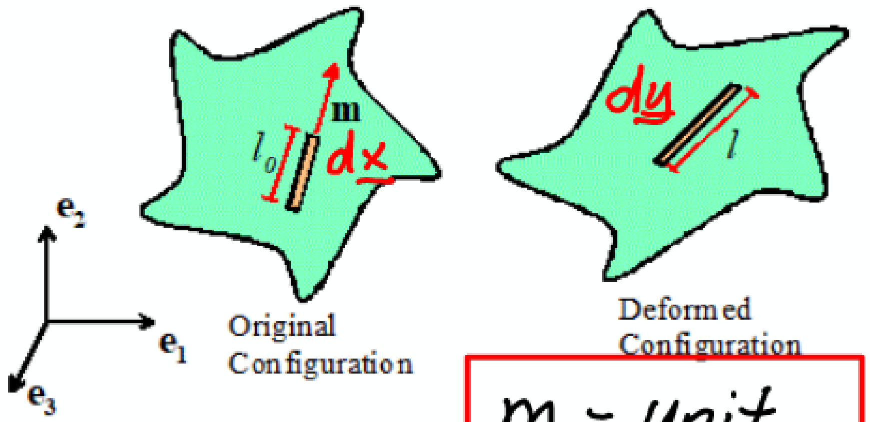
$$E_{ij} = \frac{1}{2}(F_{ki} F_{kj} - \delta_{ij})$$

Note E is symmetric $\quad E_{ij} = E_{ji}$

Significance

E quantifies length changes of infinitessimal fibers

$$\frac{\ell^2-\ell_0^2}{2\ell_0^2} = \underline{m}\cdot E\underline{m} = m_i E_{ij} m_j$$



$\boxed{\underline{m} = \text{unit vector}}$

To see this note $\quad \ell^2-\ell_0^2 = d\underline{y}\cdot d\underline{y} - d\underline{x}\cdot d\underline{x}$

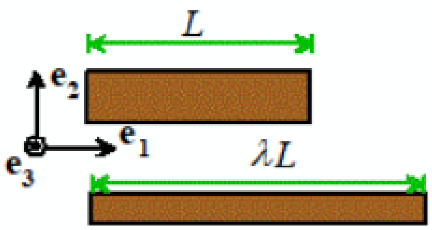
also $\quad d\underline{x} = \ell_0\,\underline{m} \qquad d\underline{y} = F\,d\underline{x}$

$$\Rightarrow \ell^2-\ell_0^2 = \underbrace{F_{ki}\,\ell_0 m_i}_{dy_k}\; F_{kj}\,\ell_0 m_j - \ell_0 m_i \underbrace{\ell_0 m_i}_{dx_i}$$

$$\Rightarrow \frac{\ell^2-\ell_0^2}{2\ell_0^2} = m_i \underbrace{\frac{1}{2}\left\{ F_{ki}F_{kj} - \delta_{ij} \right\}}_{E_{ij}} m_j$$

Example: A volume preserving stretch is described by the mapping

$$y_1 = \lambda x_1$$
$$y_2 = x_2 / \sqrt{\lambda}$$
$$y_3 = x_3 / \sqrt{\lambda}$$



Formula

$$F = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \text{etc} & \end{bmatrix}$$

Find **F** and **E**

Hence $F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix}$

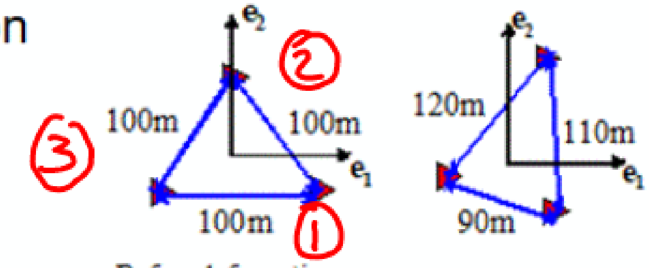
Note $\det(F) = 1$

$\Rightarrow$ Vol preserved

$$E = \frac{1}{2}\left(F^T F - I\right) = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 1/\lambda - 1 & 0 \\ 0 & 0 & 1/\lambda - 1 \end{bmatrix}$$

Example: Distances between 3 survey stations on a glacier are shown.

Find **E**



Before deformation After deformation

Formula

$$\frac{\ell^2 - \ell_0^2}{2\ell_0^2} = \underline{m} \cdot \underline{E}\,\underline{m}$$

Apply formula to 3 sides

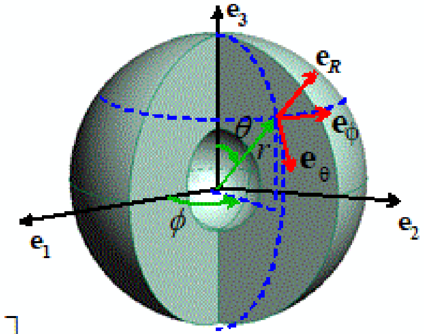
(1) $$\frac{90^2 - 100^2}{2 \times 100^2} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{11} = \frac{-19}{200}$$

(2) $$\frac{110^2 - 100^2}{2 \times 100^2} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix} = \frac{E_{11} + 3E_{22} - 2\sqrt{3}E_{12}}{4}$$

(3) $$\frac{120^2 - 100^2}{2 \cdot 100^2} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix} \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} = \frac{E_{11} + 3E_{22} + 2\sqrt{3}E_{12}}{4}$$

3 equations : Solve $E_{11} = \frac{-19}{200}$ $E_{22} = \frac{149}{600}$ $E_{12} = \frac{23\sqrt{3}}{600}$

Example: An incompressible spherical shell is inflated.
A point that starts at $\mathbf{x} = R\mathbf{e}_R$ moves to $\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3} \mathbf{e}_R$



Find **F** (in polar coordinates)

$F = \nabla y$

$$\nabla \mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R} & \frac{1}{R\sin\theta}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R} \\ \frac{\partial v_\theta}{\partial R} & \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R} & \frac{1}{R\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \cot\theta\frac{v_\phi}{R} \\ \frac{\partial v_\phi}{\partial R} & \frac{1}{R}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta\frac{v_\theta}{R} + \frac{v_R}{R} \end{bmatrix}$$

(From notes)

Here $y_R = (R^3 + a^3 - A^3)^{1/3}$ $\quad y_\theta = y_\phi = 0$

Hence $\partial y_R / \partial R = \dfrac{R^2}{(R^3 + a^3 - A^3)^{2/3}} = \dfrac{R^2}{y_R^2}$

$$F = \begin{bmatrix} R^2/y_R^2 & 0 & 0 \\ 0 & y_R/R & 0 \\ & 0 & y_R/R \end{bmatrix}$$

Note $\det(F) = 1$
$\Rightarrow$ Vol preserved

4.5 Infinitessimal Strain Tensor

Definition $\varepsilon = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)$ $\qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$[\varepsilon] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \text{Sym} & & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

ε is an approximate deformation measure used when deformation & rotation are small

(all components of $\nabla \underline{u}$ are $\ll 1$)

$$\boxed{\text{For } \quad \nabla \underline{u} \ll 1 \qquad \varepsilon \approx E}$$

To see this note $E = \frac{1}{2}\left(F^T F - I \right)$

Recall $F = I + \nabla \underline{u}$

Hence $E = \frac{1}{2}\left\{ (I + \nabla \underline{u})^T (I + \nabla \underline{u}) - I \right\}$

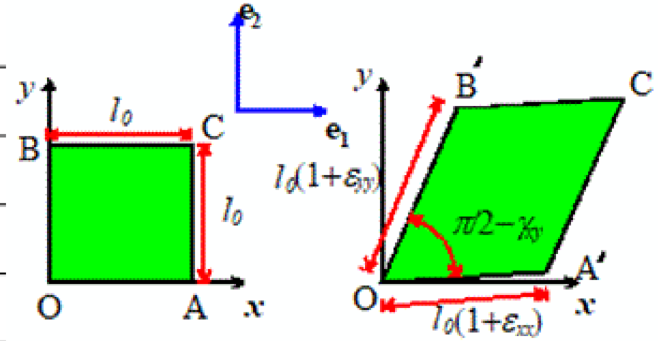
$$= \frac{1}{2}\left\{ I + \nabla \underline{u} + \nabla \underline{u}^T + (\nabla \underline{u})^T \nabla \underline{u} - I \right\}$$

$$= \varepsilon + \frac{1}{2} (\nabla \underline{u})^T \nabla \underline{u}$$

Neglect

Properties of ε

The components of ε quantify length and angle changes of infinitesimal cube



Eg in 2d $\quad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}$

$\varepsilon_{11} = \varepsilon_{xx}$
$\varepsilon_{22} = \varepsilon_{yy}$
$\varepsilon_{12} = \gamma_{xy}/2$

NB : "Engineering" shear strain $\quad \gamma_{xy} = 2\varepsilon_{12}$

ABAQUS reports "engineering" shear strains – off diagonals are all doubled.