Review

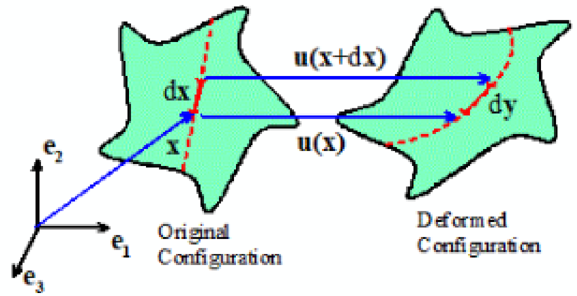
Deformation Mapping: $\mathbf{y}(\mathbf{x},t)$

Displacement Vector: $\mathbf{u}(\mathbf{x},t) = \mathbf{y}(\mathbf{x},t) - \mathbf{x}$

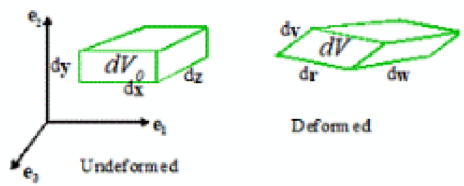
Deformation Gradient: $\mathbf{F} = \nabla\mathbf{y} = \nabla\mathbf{u} + \mathbf{I}$

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} + \delta_{ij}$$

$$d\mathbf{y} = \mathbf{F}\cdot d\mathbf{x} \qquad dy_i = F_{ij}dx_j$$



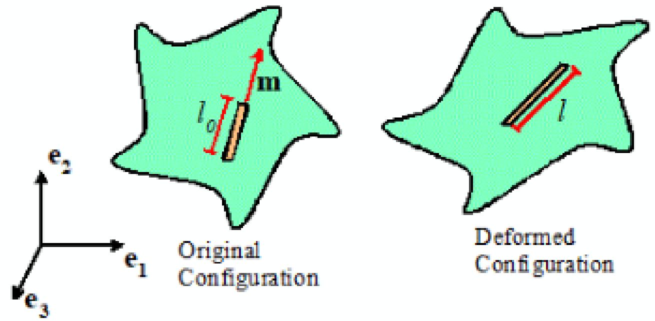
Jacobian: $J = \det(\mathbf{F}) \quad dV = JdV_0$



Lagrange Strain: $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$

$$E_{ij} = \frac{1}{2}(F_{ki}F_{kj} - \delta_{ij})$$

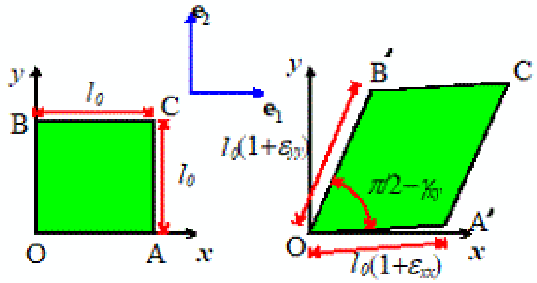
$$\frac{l^2 - l_0^2}{2l_0^2} = \mathbf{m}\cdot\mathbf{E}\mathbf{m} = m_iE_{ij}m_j$$

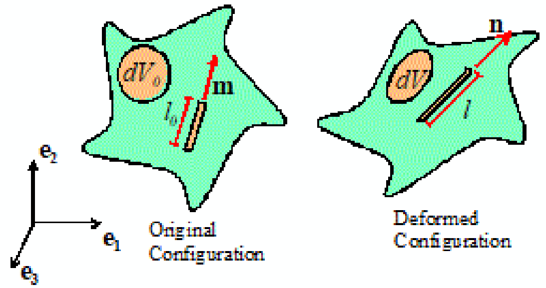


Review

Infinitesimal strain: $\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Properties:
(1) Approximate strain measure used only for small deformation
(2) For small strains $\boldsymbol{\varepsilon} \approx \mathbf{E}$
(3) Components quantify length and angle changes of unit cube





Additional properties of ε

(4) Length changes

$$\underline{m} \cdot \varepsilon \, \underline{m} \approx \frac{l - l_0}{l_0} = \frac{\delta l}{l_0}$$ uniaxial engineering strain of fiber

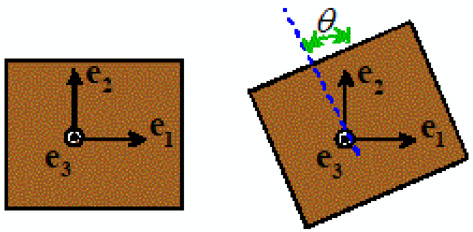
(5) Volume Changes

Vol change

$$\mathrm{trace}(\varepsilon) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \underline{u} \approx \frac{dV - dV_0}{dV_0}$$

Example: A 2D rigid rotation is described by the mapping

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Find $\boldsymbol{\varepsilon}$ and **E**

Formulas

$$\varepsilon = [\nabla \underline{u} + (\nabla \underline{u})^T]/2$$

$$E = (F^T F - I)/2$$

$$F = I + \nabla \underline{u}$$

$$u_1 = (\cos\theta - 1)x_1 + \sin\theta\, x_2$$

$$u_2 = -\sin\theta\, x_1 + (\cos\theta - 1)x_2$$

$$\nabla \underline{u} = \begin{bmatrix} \cos\theta - 1 & \sin\theta \\ -\sin\theta & \cos\theta - 1 \end{bmatrix} \Rightarrow \varepsilon = \begin{bmatrix} \cos\theta - 1 & 0 \\ 0 & \cos\theta - 1 \end{bmatrix}$$

$$\varepsilon \approx 0 \quad \text{if} \quad \theta << 1 \quad (\cos\theta \approx 1)$$

$$F = I + \nabla \underline{u} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$E = \frac{1}{2} \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

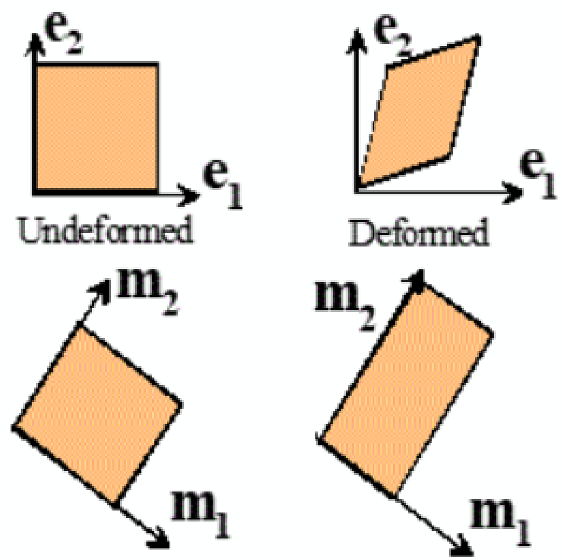
$$= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ zero for arbitrary θ

4.6 Principal Strains

Note E & ε are symmetric
$\Rightarrow$ we can always find a basis that makes components a diagonal matrix

Infinitessimal strain

$$[\varepsilon^{(\underline{m})}] = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}$$



e_i = eigenvalues of $[\varepsilon]$

$\underline{m}_i$ = eigenvectors $\{\underline{m}_1, \underline{m}_2, \underline{m}_3\}$

Physical Interpretation :

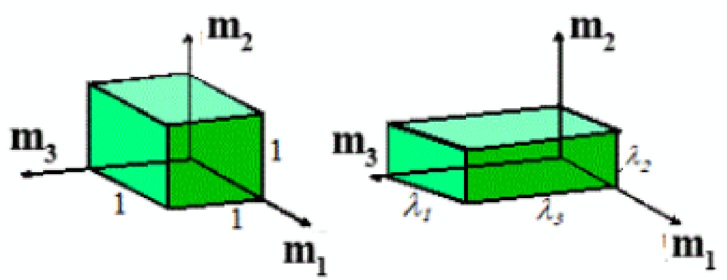
In special basis off diags are zero ⇒ no angle changes of unit cube with faces perpendicular to $\underline{m}_i$ during deformation
- cube deforms into rectangular prism

Lagrange Strain

① Recall $\underline{m} \cdot E \underline{m} = \dfrac{\ell^2 - \ell_0^2}{2\ell_0^2} = \frac{1}{2}(\lambda^2 - 1)$ $\lambda = \dfrac{\ell}{\ell_0}$ "stretch ratio"

② If $\underline{m}_i$ is an eigenvector $E\underline{m}_i = \underbrace{\frac{1}{2}(\lambda_i^2 - 1)}_{\text{eigenvalue}}\underline{m}_i$

Hence
$$[E^{(m)}] = \begin{bmatrix} (\lambda_1^2-1)/2 & 0 & 0 \\ 0 & (\lambda_2^2-1)/2 & 0 \\ 0 & 0 & (\lambda_3^2-1)/3 \end{bmatrix}$$

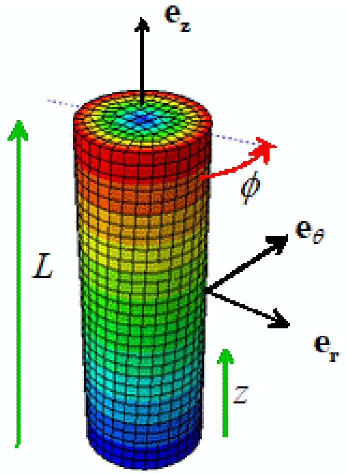


Example: Torsion of a cylinder:

$$\mathbf{u} = \frac{\phi}{L} r z \mathbf{e}_\theta$$

Find $\boldsymbol{\varepsilon}$ and **E**

Find principal (infinitesimal) strains



Cylindrical-polar coords:

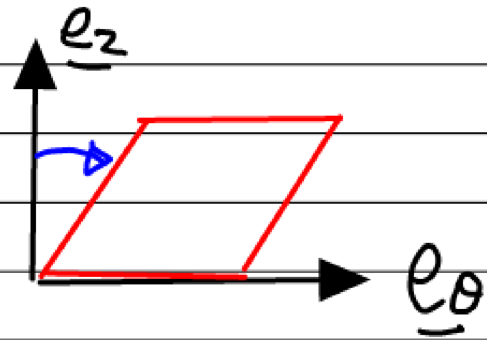
$$\nabla \mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Here $u_r = 0$ $\quad u_\theta = \frac{\phi}{L} r z$ $\quad u_z = 0$

Hence $\nabla \underline{u} = \frac{\phi}{L}\begin{bmatrix} 0 & -z & 0 \\ z & 0 & r \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \varepsilon = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T) = \frac{\phi}{2L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & r & 0 \end{bmatrix}$

$\varepsilon_{z\theta} = \varepsilon_{\theta z} = \frac{\phi r}{2L}$

$2\varepsilon_{z\theta} = \frac{\phi r}{L}$



$$E = \varepsilon + \frac{1}{2}(\nabla \underline{u})^T(\nabla \underline{u})$$

$$= \varepsilon + \frac{1}{2}\frac{\phi^2}{L^2}\begin{bmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & r & 0 \end{bmatrix}\begin{bmatrix} 0 & -z & 0 \\ z & 0 & r \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\phi}{2L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r \\ 0 & r & 0 \end{bmatrix} + \frac{\phi^2}{2L^2}\begin{bmatrix} z^2 & 0 & rz \\ 0 & z^2 & 0 \\ rz & 0 & r^2 \end{bmatrix}$$

Principal infinitessimal strain $\det(\varepsilon - e_i I) = 0$

$$\det\begin{bmatrix} -e_i & 0 & 0 \\ 0 & -e_i & \frac{r\phi}{2L} \\ 0 & \frac{\phi r}{2L} & -e_i \end{bmatrix} = 0 \Rightarrow -e_i\left(e_i^2 - \frac{r^2\phi^2}{4L^2}\right) = 0$$

$$\boxed{e_1 = 0 \qquad e_2 = -r\phi/2L \qquad e_3 = +r\phi/2L}$$

```
syms r z phi L
eps = [0, 0, 0;...
       0, 0, r*phi/L/2;...
       0, r*phi/L/2,0]
[dirs,pvals] = eig(eps)
```

eps =

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\varphi r}{2L} \\ 0 & \frac{\varphi r}{2L} & 0 \end{pmatrix}$$

dirs =

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}_{\text{cols are eigenvectors (not unit vecs)}}$$

pvals =

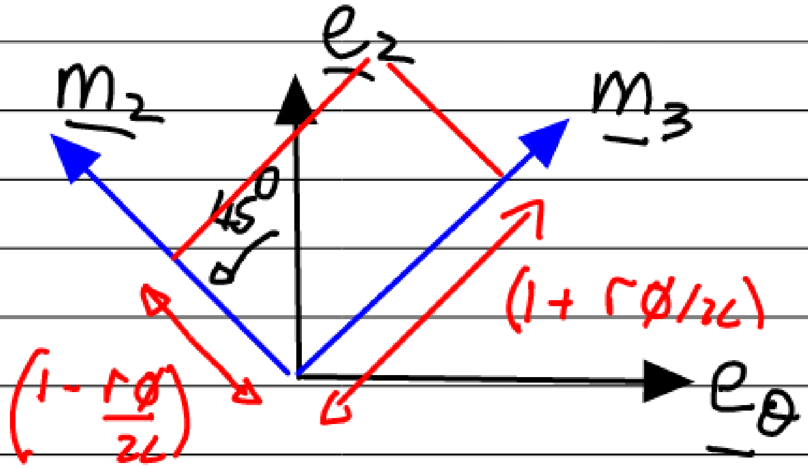
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\varphi r}{2L} & 0 \\ 0 & 0 & \frac{\varphi r}{2L} \end{pmatrix}$$

Eigenvals

Hence $\underline{m}_1 = \underline{e}_r$

$\underline{m}_2 = (-\underline{e}_\theta + \underline{e}_z)/\sqrt{2}$

$\underline{m}_3 = (\underline{e}_\theta + \underline{e}_z)/\sqrt{2}$



4.7 Compatibility conditions for infinitesimal strains

Problem: Given ε_{11} ε_{22} ε_{12} as functions of x_1, x_2

Find u_1 u_2

We have to integrate strains somehow

Strains can be integrated only if they satisfy "compatibility condition"

Recall $\varepsilon_{11} = \dfrac{\partial u_1}{\partial x_1}$ $\quad \varepsilon_{12} = \dfrac{1}{2}\left\{ \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} \right\}$

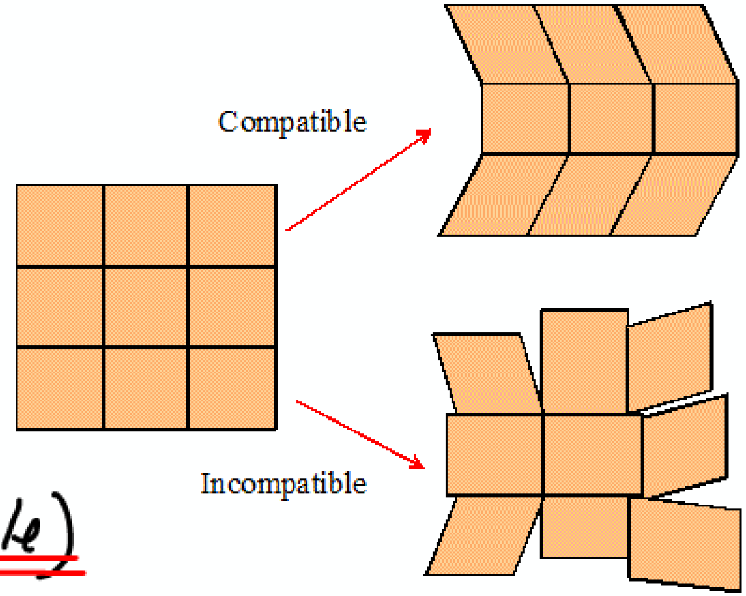
$\varepsilon_{22} = \dfrac{\partial u_2}{\partial x_2}$

$$\boxed{\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0}$$

Compatibility condition

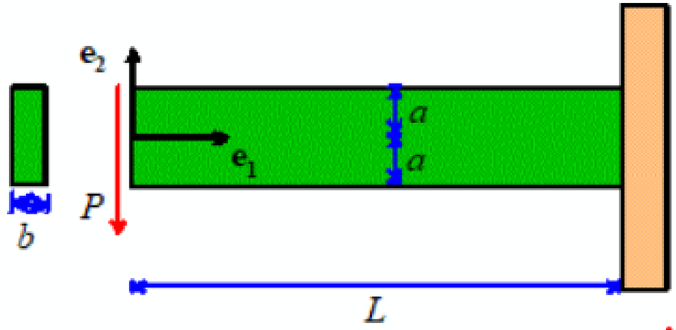
Physical interpretation

Strains must be compatible to "fit together" and give a continuous displacement



Integrating strains (consider example)

$$\varepsilon_{11} = 2Cx_1x_2 \quad \varepsilon_{22} = -2\nu Cx_1x_2 \quad \varepsilon_{12} = (1+\nu)C\left(a^2 - x_2^2\right), \; C = \frac{3P}{4Ea^3b}$$



Find u_1, u_2

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = 2Cx_1x_2 \Rightarrow u_1 = Cx_1^2x_2 + Cf(x_2)$$

integration "consts"

$$\varepsilon_{22} = \partial u_2/\partial x_2 = -2\nu Cx_1x_2 \Rightarrow u_2 = -\nu x_1x_2^2C + Cg(x_1)$$

Finally $2\varepsilon_{12} = \dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1} = 2(1+\nu)C(a^2 - x_2^2)$

Substitute : $C\left\{ x_1^2 + \dfrac{\partial f}{\partial x_2} - \nu x_2^2 + \dfrac{\partial g}{\partial x_1} \right\} = 2(1+\nu)C(a^2 - x_2^2)$

Collect terms :

$$\underbrace{C\left\{ x_1^2 + \frac{\partial g}{\partial x_1} \right\}}_{\substack{\text{constant} \\ \omega}} - \underbrace{C\left\{ \nu x_2^2 + 2(1+\nu)(a^2 - x_2^2) - \frac{\partial f}{\partial x_2} \right\}}_{\omega \quad \text{(also constant)}} = 0$$

Hence $C\dfrac{\partial g}{\partial x_1} = \omega - Cx_1^2 \Rightarrow Cg = \omega x_1 - \dfrac{1}{3}Cx_1^3 + A$

$C\dfrac{\partial f}{\partial x_2} = -\omega + C\nu x_2^2 + 2(1+\nu)C(a^2 - x_2^2)$

Integration constants

$Cf = -\omega x_2 + C\nu x_2^3/3 + 2(1+\nu)(a^2 x_2 - x_2^3/3) + B$

Substitute back into u_1, u_2

$$u_1 = Cx_1^2 x_2 - \frac{C}{3}(2+\nu)x_2^3 + 2(1+\nu)Ca^2 x_2 - \omega x_2 + B$$

$$u_2 = -\nu C x_1 x_2^2 - \frac{C}{3} x_1^3 + \omega x_1 + A$$

Note $\left.\begin{aligned} u_1 &= B - \omega x_2 \\ u_2 &= A + \omega x_1 \end{aligned}\right\}$ unknown small rotation about e_3 thru ω and translation (A, B)

If displacements are known @ 3 points can find A, B, ω

```
syms x1 x2 C a nu
syms f(x2) g(x1)
syms omega
e11 = 2*C*x1*x2;
e22 = -2*nu*C*x1*x2;
e12 = (1+nu)*C*(a^2-x2^2);
u1 = int(e11,x1) + C*f(x2);
u2 = int(e22,x2) + C*g(x1);
eq = diff(u1,x2) + diff(u2,x1) - 2*e12
terms = children(eq)
de1 = terms(2)+terms(3)==omega;
de2 = terms(1) + terms(4) + terms(5)==-omega;
gsol = dsolve(de1,symvar('g(x1)'))
fsol = dsolve(de2,symvar('f(x2)'))
u1 = simplify(subs(u1,f(x2),fsol))
u2 = simplify(subs(u2,g(x1),gsol))
```

u1 =

$$C\,x_2\,x_1^2 + C\,\left(\left(-\frac{\nu}{3}-\frac{2}{3}\right)x_2^3 + \left(2\,a^2\,\nu - \frac{\omega}{C} + 2\,a^2\right)x_2 + C_2\right)$$

u2 =

$$C\,\left(C_1 - \frac{x_1^3}{3} + \frac{\omega\,x_1}{C}\right) - C\,\nu\,x_1\,x_2^2$$