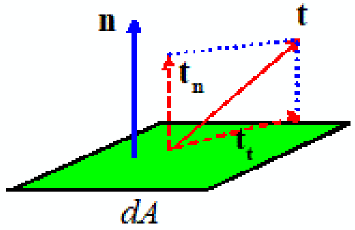
5 Describing Internal Forces in Solids

5.1 External Forces

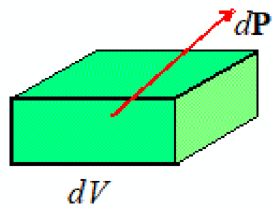


Traction Vector $\quad \underline{t} = \lim_{dA \to 0} \dfrac{d\underline{P}}{dA}$

Normal traction $\quad t_n = \underline{t} \cdot \underline{n}$

Tangential $\quad \underline{t}_t = \underline{t} - t_n \underline{n}$

Total force on surface : $\underline{P} = \int_S \underline{t}\, dA$

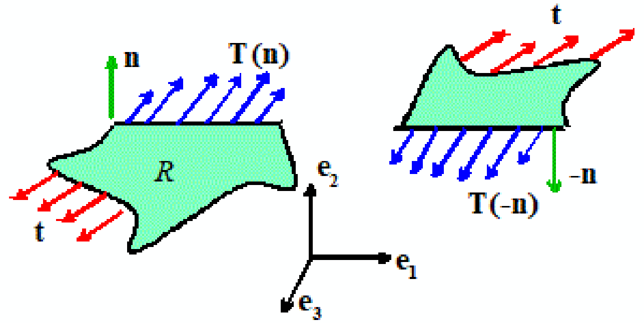


Body force vector (force per unit mass)

$\underline{b} = \lim_{dV \to 0} \dfrac{d\underline{P}}{\rho\, dV}$ $\qquad \rho$: mass density

Resultant body force $\underline{P} = \int_V \rho \underline{b} \, dV$

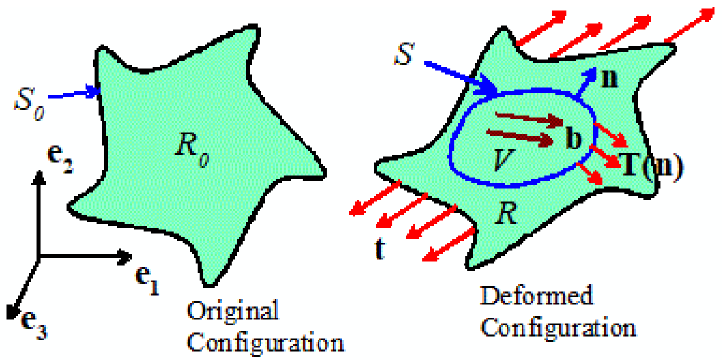
5.2 Internal traction vector



Internal traction $\underline{T}(\underline{n}) = \lim_{dA \to 0} \dfrac{d\underline{P}(\underline{n})}{dA}$

$\underline{n}$: unit vector

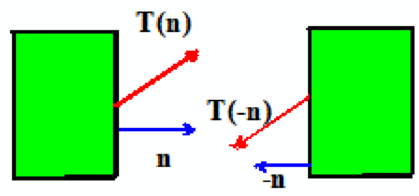
Total force acting on a sub-volume of solid



$$\underline{P} = \int_S \underline{T}(\underline{n}) \, dA + \int_V \rho \underline{b} \, dV$$

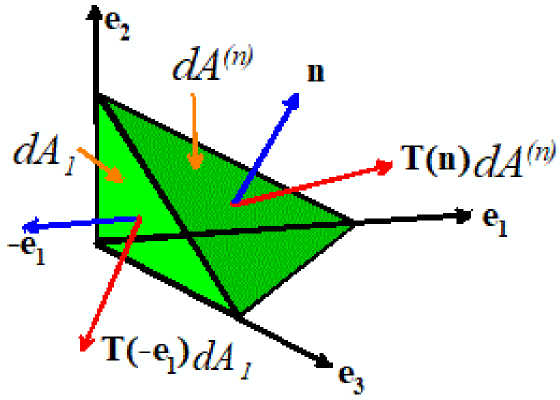
Properties of $\underline{T}(\underline{n})$

① Tractions on adjacent planes are equal & opposite



$$\underline{T}(-\underline{n}) = -\underline{T}(\underline{n})$$

② Tractions on surfaces passing through a point are related

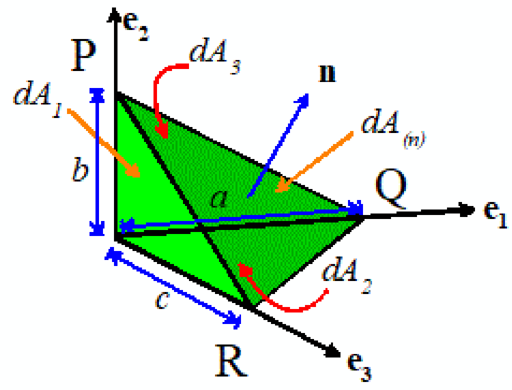


$$\underline{T}(\underline{n}) = \underline{T}(\underline{e}_1)n_1 + \underline{T}(\underline{e}_2)n_2 + \underline{T}(\underline{e}_3)n_3$$

(Consequence of $\underline{F} = m\underline{a}$)

$$\underline{n} = n_1\underline{e}_1 + n_2\underline{e}_2 + n_3\underline{e}_3$$

Proof



① Show first that

$$dA_n \underline{n} = dA_1 \underline{e}_1 + dA_2 \underline{e}_2 + dA_3 \underline{e}_3$$

Note $2 dA_n \underline{n} = \vec{PR} \times \vec{PQ}$

$$= (c\underline{e}_3 - b\underline{e}_2) \times (a\underline{e}_1 - b\underline{e}_2)$$

$$= cb\underline{e}_1 + ca\underline{e}_2 + ab\underline{e}_3$$

$$= 2(dA_1 \underline{e}_1 + dA_2 \underline{e}_2 + dA_3 \underline{e}_3) \qquad \text{QED!}$$

Hence $\underline{n} = \underbrace{\frac{dA_1}{dA_n}}_{n_1} \underline{e}_1 + \underbrace{\frac{dA_2}{dA_n}}_{n_2} \underline{e}_2 + \underbrace{\frac{dA_3}{dA_n}}_{n_3} \underline{e}_3$

② Now $\underline{F} = m\underline{a}$ for tetrahedron

$$\underline{T}(\underline{n})\,dA_n + \underline{T}(-\underline{e}_1)\,dA_1 + \underline{T}(-\underline{e}_2)\,dA + \underline{T}(-\underline{e}_3)\,dA_3$$

$$+ \rho\underline{b}\,dV = \rho\,dV\,\underline{a}$$

Divide through by dA_n ; note $\lim_{dV\to 0} \frac{dV}{dA_n} \to 0$

$$\underline{T}(\underline{n}) - \underline{T}(\underline{e}_1)\underbrace{\frac{dA_1}{dA_n}}_{n_1} - \underline{T}(\underline{e}_2)\underbrace{\frac{dA_2}{dA_n}}_{n_2} - \underline{T}(\underline{e}_3)\underbrace{\frac{dA_3}{dA_n}}_{n_3} = 0$$

QED !

5·3 Cauchy Stress Tensor

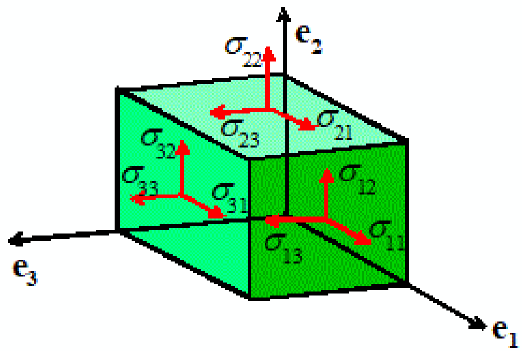
Define $\sigma_{ij} = T_j(\underline{e}_i)$ (9 numbers)

Then $T_j(\underline{n}) = n_i \sigma_{ij}$ $\quad \underline{T}(\underline{n}) = \underline{n}\,\sigma$

Note σ maps $\underline{n}$ onto $\underline{T}$ $\Rightarrow$ tensor!

Physical Significance

σ_{ij} = force per unit deformed area acting in j direction on plane with normal in i direction



Health warning: some people use transpose $\underline{T} = \sigma\,\underline{n}$
– fortunately $\sigma_{ij} = \sigma_{ji}$ (symmetric)

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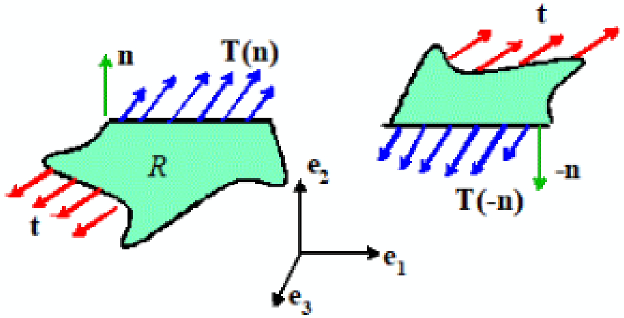
Other stress measures

Cauchy stress is 'True Stress' (Force per unit deformed area

$$T_j(\mathbf{n}) = n_i \sigma_{ij} \qquad \mathbf{T} = \mathbf{n} \cdot \boldsymbol{\sigma}$$

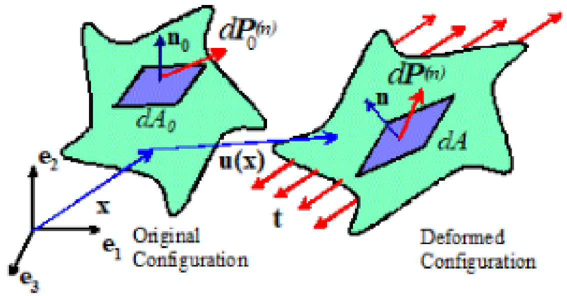
True stress is symmetric



There are also several types of 'Nominal Stress' (Force per unit undeformed area)

Deformation gradient $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} \qquad F_{ij} = \delta_{ij} + \dfrac{\partial u_i}{\partial x_j}$

Jacobian $J = \det(\mathbf{F})$



Kirchhoff stress $\boldsymbol{\tau} = J\boldsymbol{\sigma}$

Nominal (First Piola-Kirchhoff Stress) $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \qquad S_{ij} = JF_{ik}^{-1}\sigma_{kj}$

Has property that $d\mathbf{P} = dA_0 \mathbf{n}_0 \cdot \mathbf{S} = dA\, \mathbf{n} \cdot \boldsymbol{\sigma}$ Unsymmetric – some texts use transpose

Material (Second Piola-Kirchhoff Stress) $\boldsymbol{\Sigma} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} \qquad \Sigma_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$

Has property that $d\mathbf{P}_0 = \mathbf{F}^{-1} d\mathbf{P} = dA_0 \mathbf{n}_0 \cdot \boldsymbol{\Sigma}$ (Weird but useful for material models)

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Example: Consider a hydrostatic stress $\sigma_{ij} = -p\delta_{ij}$

Show that the traction acts normal to any internal plane in the solid (fluid?) and find its magnitude

$$\sigma = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

Formula $T_i(n_j) = n_j \sigma_{ji} = n_j(-p\delta_{ij})$

$T_i(n_j) = -p\, n_i$ $\quad \underline{T}(\underline{n}) = \underset{\text{mag}}{-p}\ \underset{\text{Dir}}{\underline{n}}$

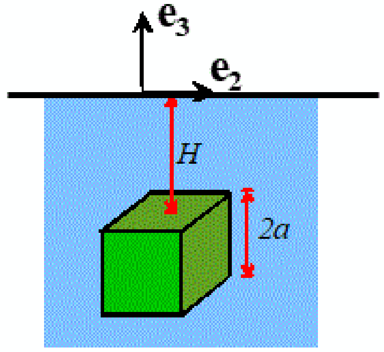
Direction of $\underline{T}$ is parallel to $\underline{n}$

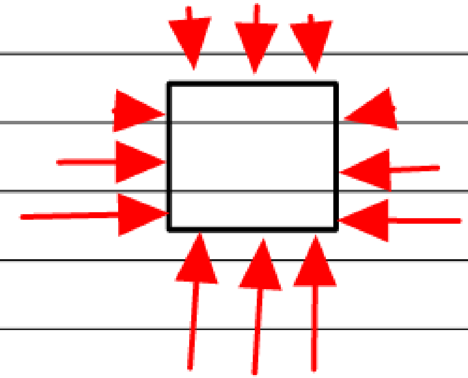
(hydrostatic pressure)

Example: A cube is immersed in fluid with stress tensor

$$\sigma_{ij} = \rho g x_3 \delta_{ij}$$

Find the resultant force on the cube





Forces on sides with normals $\underline{e}_1$, $\underline{e}_2$ are equal & opposite – no horizontal force

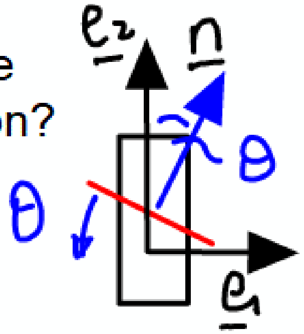
Force on top face $\qquad \rho g H (2a)^2 \, (-\underline{e}_3)$

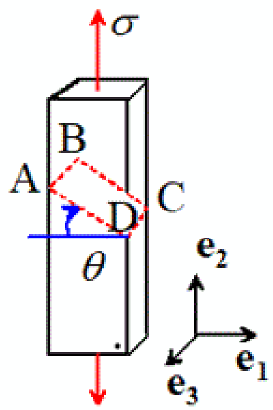
on bottom face $\qquad \rho g (H+2a)(2a)^2 \, (\underline{e}_3)$

Total : $\underline{P} = \underbrace{(2a)^3 \rho g}_{\text{weight of displaced fluid}} \; \underline{e}_3$

Example: Find the tractions on the plane ABCD

What is the inclination of the plane with the greatest tangential traction?





Stress tensor

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Formula $\underline{T} = \underline{n}\,\sigma$ Here $\underline{n} = [\sin\theta \quad \cos\theta \quad 0]$

$$\underline{T} = [\sin\theta \quad \cos\theta \quad 0] \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma\cos\theta \\ 0 \end{bmatrix}$$

Normal traction $\underline{T}_n = \underline{T} \cdot \underline{n} = \sigma\cos^2\theta$

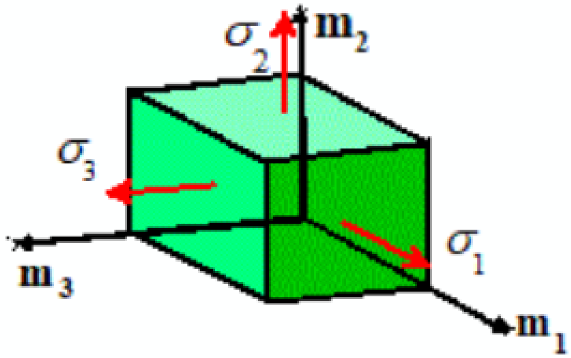
Tangential $\underline{T}_t = \underline{T} - T_n\underline{n}$ $\begin{bmatrix} 0 \\ \sigma\cos\theta \\ 0 \end{bmatrix} - \sigma\cos^2\theta \begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$

$$|\underline{T}_t| = \sigma\cos\theta\sqrt{(\sin\theta\cos\theta)^2 + (1-\cos^2\theta)^2} = \sigma\sin\theta\cos\theta$$

$$= \frac{1}{2}\sigma\sin 2\theta \qquad \max\{|\underline{T}_t|\} = \boxed{\sigma/2 \text{ for } \theta = \pi/4}$$

5.4 Principal Stresses

Note σ is symmetric



We can find a special basis in which σ is diagonal

$$[\sigma^{(\underline{m})}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

σ_i : eigenvalues of $[\sigma]$

$\{\underline{m}_1, \underline{m}_2, \underline{m}_3\}$: eigenvectors

Force per unit area on planes perpendicular to $\underline{m}_i$ is normal to surface

Some useful properties of principal stresses

① Maximum normal stress (traction) on any plane @ a point is $\max(\sigma_1, \sigma_2, \sigma_3)$

② max shear stress on any plane @ a point is

$$\tau_{max} = \frac{1}{2} \max \{ \underbrace{|\sigma_1 - \sigma_2|}_{\text{plane @ } 45^\circ \text{ to } (\underline{e}_1, \underline{e}_2)}, \underbrace{|\sigma_1 - \sigma_3|}_{\text{plane @ } 45^\circ \text{ to } (e_1, e_3)}, |\sigma_2 - \sigma_3| \}$$