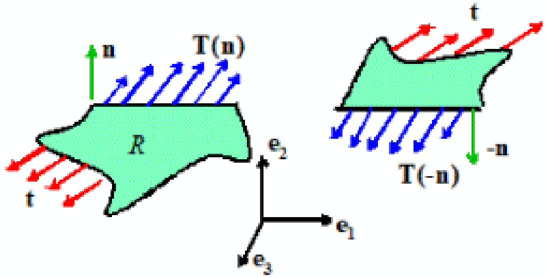
Review



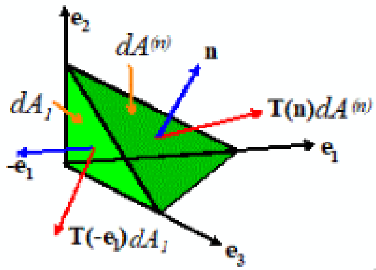
Internal Traction Vector T(n)

Quantifies force per unit area at a point on internal plane

Traction depends on direction of normal to surface

Satisfies: $\mathbf{T}(-\mathbf{n}) = -\mathbf{T}(\mathbf{n})$

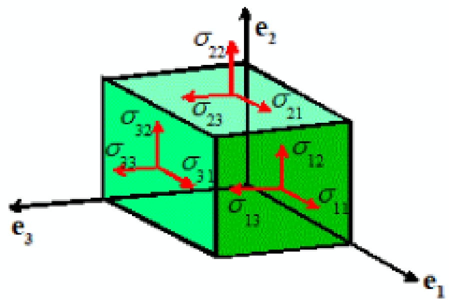
$$\mathbf{T}(\mathbf{n}) = \mathbf{T}(\mathbf{e}_1)n_1 + \mathbf{T}(\mathbf{e}_2)n_2 + \mathbf{T}(\mathbf{e}_3)n_3$$



Cauchy ("True") Stress Tensor

Definition (components): $\sigma_{ij} = T_j(\mathbf{e}_i)$

Then: $T_j(\mathbf{n}) = n_i \sigma_{ij}$ $\qquad \mathbf{T} = \mathbf{n} \cdot \boldsymbol{\sigma}$



Warning: Some texts use transpose of this definition $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$

Cauchy stress (force per unit deformed area) is symmetric $\sigma_{ij} = \sigma_{ji}$, so both are the same, but some other stresses eg nominal stress (force per unit undeformed area) are not, so be careful.

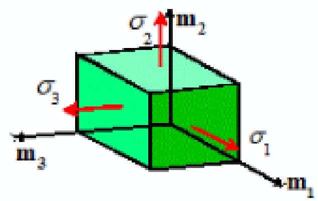
Principal Stresses

We can find a basis that makes $\boldsymbol{\sigma}$ components a diagonal matrix

$$\left[\sigma^{(\mathbf{m})}\right] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$(\sigma_1, \sigma_2, \sigma_3)$ (eigenvalues "principal stresses")

$\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ (eigenvectors)



5.5 Other stress measures

Hydrostatic stress

$$\sigma_h = \frac{1}{3}\,\mathrm{trace}(\sigma) = \frac{1}{3}\sigma_{kk} = \frac{1}{3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$$

measure of pressure

Deviatoric stress

Identity

: $S = \sigma - \sigma_h I \qquad S_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$

tensor measure of shear stress

Von-Mises "effective" stress (shear stress magnitude)

$$\sigma_e = \sqrt{\frac{3}{2} S : S} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} = \sqrt{\frac{3}{2}\left\{ S_{11}^2 + S_{12}^2 + S_{13}^2 + S_{21}^2 + \cdots \right\}}$$

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]}$$

5.6 Simple failure criteria

Brittle materials : cracks propagate if stress acting normal to crack exceeds critical value

$$\max\{\sigma_1, \sigma_2, \sigma_3\} \geq \sigma_f \Rightarrow \text{fracture}$$

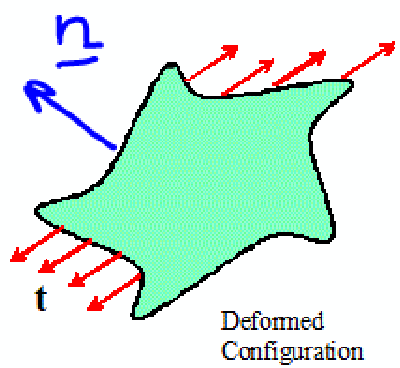
fracture stress - measure it

Yield criterion (metals)

$$\sigma_e \geq Y \Rightarrow \text{yield (permanent deformation)}$$

Yield stress (material property - measure experimentally)

5.7 Stresses at an exterior surface



Stress tensor has to be consistent with external traction

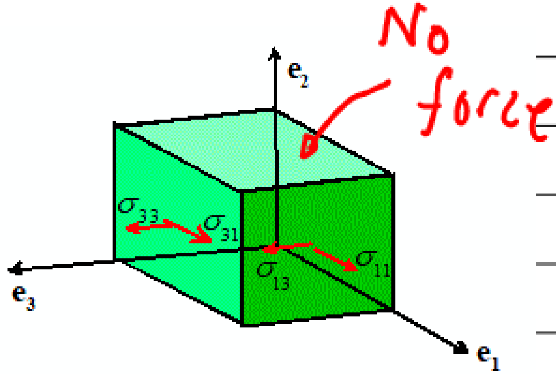
$\underline{n}$: unit vector

Hence $\quad \underline{n}\,\sigma = \underline{t}$

Example: Traction free surface

$$[0 \;\; 1 \;\; 0] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

using symmetry



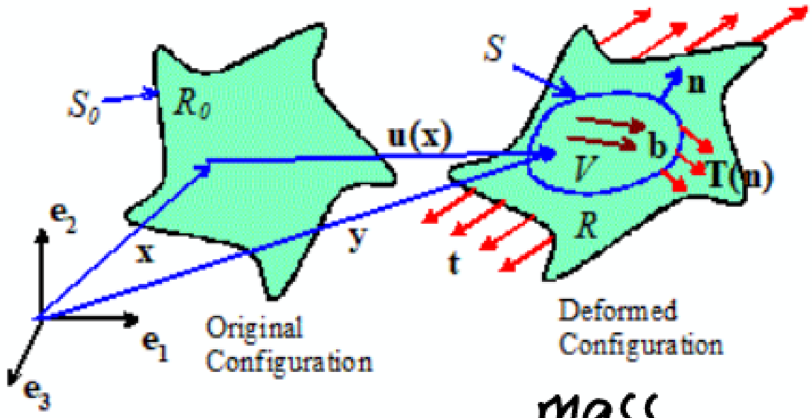
$$\Rightarrow \quad \sigma_{12} = \sigma_{21} = \sigma_{22} = \sigma_{23} = \sigma_{32} = 0$$

6 Equations of motion for solids

Assumptions:

(1) Internal forces described by Cauchy stress σ

(2) Forces on any interior sub-vol obey Newtons laws



mass density ρ

Goal: Find linear & angular momentum balance laws in terms of σ

6.1 Linear Momentum

$$\underline{F} = \frac{d}{dt}(m\underline{v}) \Rightarrow \int_S \underline{T}(\underline{n})\, dA + \int_V \rho\, \underline{b}\, dV = \frac{d}{dt}\left\{ \int_V \rho\, \underline{v}\, dV \right\}$$

Recall $\underline{n}\,\sigma = \underline{T}(\underline{n})$

Divergence Theorem $\displaystyle\int_S \underline{n}\,\sigma\, dA = \int_V \nabla_{\underline{y}} \cdot \sigma\, dV$

↑ Derivative wrt deformed position

Can show $\displaystyle\frac{d}{dt}\int_{\dot V} \rho\, \underline{v}\, d\dot V = \int_{\dot V} \rho \frac{\partial \underline{v}}{\partial t}\, d\dot V$

Hence $\displaystyle\int_{\dot V} \left\{ \nabla_{\underline{y}} \cdot \sigma + \rho\, \underline{b} - \rho \frac{\partial \underline{v}}{\partial t} \right\} d\dot V = 0$

Must be true for any $\dot V$ ⇒ integrand is zero

$$\underbrace{\nabla_y \cdot \sigma + \rho\, \underline{b}}_{\text{Force on small vol}} = \rho \frac{\partial \underline{v}}{\partial t}$$

← $m\underline{a}$ for small vol

Index notation

$$\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho \frac{\partial v_j}{\partial t}$$

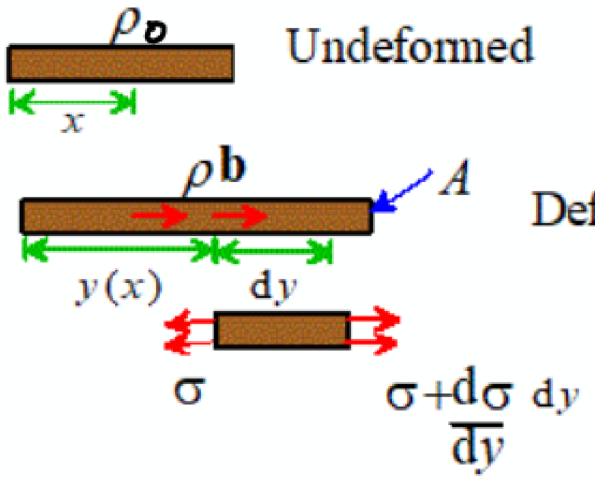
In full

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho \frac{\partial v_1}{\partial t}$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho \frac{\partial v_2}{\partial t}$$

etc ... (2 becomes 3)

Physical Significance



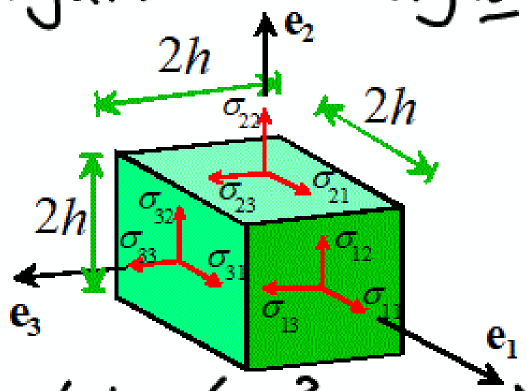
F = ma for 1D stress state

$$\left(\sigma + \frac{d\sigma}{dy}dy\right)A - \sigma A + \rho b A dy = \rho A dy \frac{\partial v}{\partial t}$$

$$\Rightarrow \frac{d\sigma}{dy} + \rho b = \rho \frac{\partial v}{\partial t}$$

6.2 Angular Momentum

Angular velocity $\underline{\omega}$



$$\Sigma \underline{r} \times \underline{F} = \frac{d}{dt}\{\underline{h}\}$$

angular momentum

$$\Rightarrow 2h(\sigma_{23} - \sigma_{32})(2h)^2 \underline{e}_1 + 2h(\sigma_{31} - \sigma_{13})(2h)^2 \underline{e}_2 + 2h(\sigma_{12} - \sigma_{21})(2h)^2 \underline{e}_3 = \frac{d}{dt}\left\{\frac{1}{6}\rho(2h)^3(2h)^2 \underline{\omega}\right\}$$

Let $h \to 0$

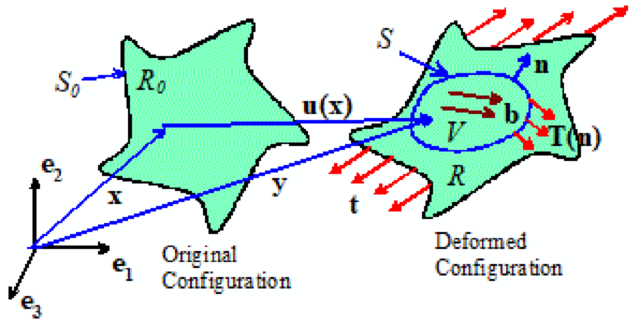
$$\Rightarrow (\sigma_{23} - \sigma_{32})\,\underline{e}_1 + (\sigma_{31} - \sigma_{13})\,\underline{e}_2 + (\sigma_{12} - \sigma_{21})\,\underline{e}_3 = \underline{0}$$

$\Rightarrow \sigma_{ij} = \sigma_{ji}$ Stress is symmetric

6.3 Approximate expression for linear momentum balance for small deformations

Exact $$\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho \frac{\partial v_j}{\partial t}$$

[NLGEOM on]



Approximate (small deformation) $$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \frac{\partial v_j}{\partial t}$$ [NLGEOM off]

initial position

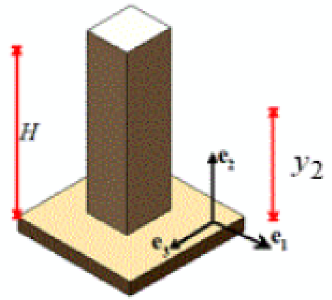
mass per unit undeformed vol

Example: The figure shows a column with mass density ρ

The top and side faces have no external traction acting on them

Show that the stress state $\sigma_{22} = -\rho g(H - y_2)$

satisfies static equilibrium and boundary conditions



Body force

$\underset{\sim}{b} = -g \, \underset{\sim}{e}_2$

(gravity)

Boundary conditions $\underset{\sim}{n} \, \sigma = \underset{\sim}{0}$ on top & sides

top $y_2 = H \Rightarrow \sigma = 0 \Rightarrow \underset{\sim}{n} \, \sigma = 0$ ✓

Sides : $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

Also $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

Linear momentum

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho \frac{dv_1}{dt}$$ $0 = 0$ ✓

(2) $$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho \frac{dv_2}{dt}$$ $\Leftarrow$ Not trivially satisfied

$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho \underbrace{\frac{dv_3}{dt}}_{\text{all } 0 \text{ (static)}}$$ $0 = 0$ ✓

(2) $$\frac{\partial \sigma_{22}}{\partial y_2} + \rho b_2 = \frac{\partial}{\partial y_2}\left(-\rho g \{H - y_2\}\right) - \rho g = 0$$ ✓

Example: Does stress field $\sigma_{ij} = \frac{-3P_k y_k y_i y_j}{4\pi R^5}$ $R = \sqrt{y_k y_k}$ satisfy equilibrium with no body forces?

Check $\frac{\partial \sigma_{ij}}{\partial y_i} = 0$?

Recall $\frac{\partial y_i}{\partial y_j} = \delta_{ij}$ $\frac{\partial R}{\partial y_i} = \frac{y_i}{R}$ $\delta_{ij} y_j = y_i$

Hence
$$\frac{\partial \sigma_{ij}}{\partial y_i} = -\frac{3P_k}{4\pi}\left\{ \frac{\overbrace{\delta_{ik}\, y_i}^{y_k} y_j}{R^5} + \frac{y_k \cancelto{3}{\delta_{ii}} y_j}{R^5} + \frac{y_k \overbrace{y_i\, \delta_{ij}}^{y_j}}{R^5} - \frac{5}{R^6} \overbrace{\frac{y_i}{R} y_k y_i}^{R^2} y_j \right\}$$

$= 0$