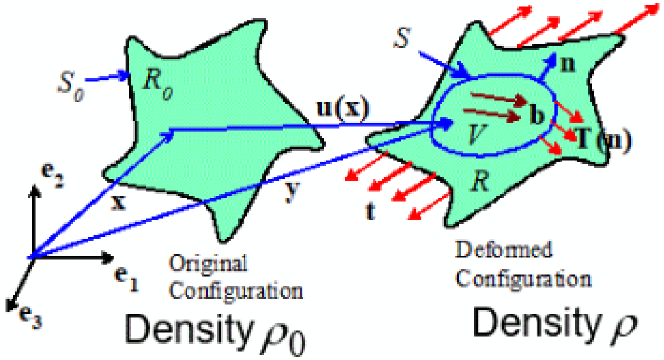
Review

Equations of motion for deformable solids

Assumptions

Internal forces characterized by Cauchy stress: σ_{ij}

Any internal volume in solid obeys Newton's laws



Linear momentum $$\mathbf{F} = \frac{d}{dt}\{m\mathbf{v}\} \Rightarrow \nabla_{\mathbf{y}} \cdot \sigma + \rho \mathbf{b} = \rho \frac{\partial \mathbf{v}}{\partial t}$$

$$\frac{\partial \sigma_{ij}}{\partial y_i} + \rho b_j = \rho \frac{\partial v_j}{\partial t}$$

$$\frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + \rho b_1 = \rho \frac{dv_1}{dt}$$

$$\frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + \rho b_2 = \rho \frac{dv_2}{dt}$$

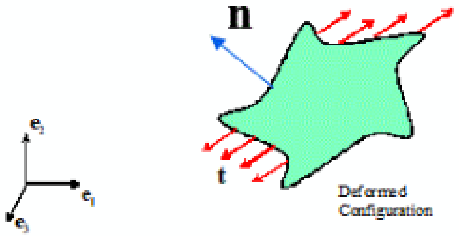
$$\frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + \rho b_3 = \rho \frac{dv_3}{dt}$$

For small deformations OK to neglect difference between $\mathbf{y}$ and $\mathbf{x}$ $$\frac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j \approx \rho_0 \frac{\partial v_j}{\partial t}$$

Angular momentum $$\mathbf{r} \times \mathbf{F} = \frac{d}{dt}\{\mathbf{r} \times m\mathbf{v}\} \Rightarrow \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \qquad \sigma_{ij} = \sigma_{ji}$$

Boundary conditions for stress at an exterior surface

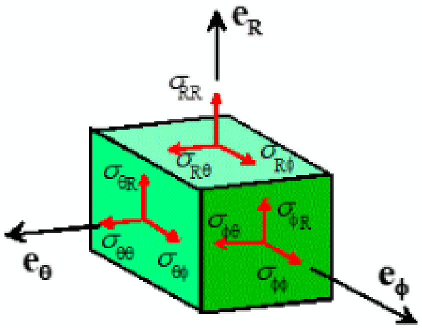
$$\mathbf{n}\boldsymbol{\sigma} = \mathbf{t} \qquad n_i \sigma_{ij} = t_j$$

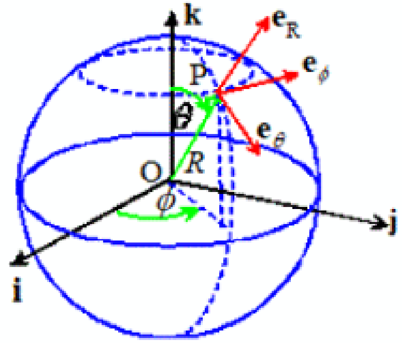


Other forms for equations of motion

Spherical-polar coordinates

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{RR} & \sigma_{R\theta} & \sigma_{R\phi} \\ \sigma_{\theta R} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi R} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

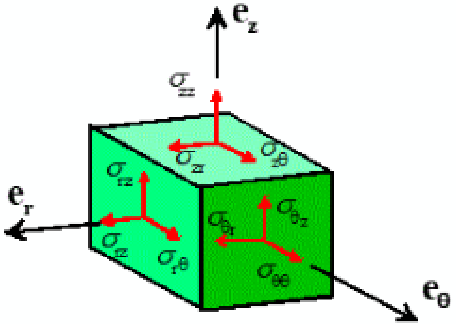


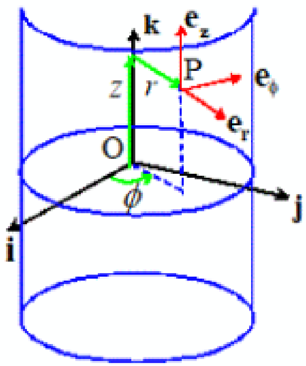


$$\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt}\equiv\begin{bmatrix} \frac{\partial\sigma_{RR}}{\partial R}+2\frac{\sigma_{RR}}{R}+\frac{1}{R}\frac{\partial\sigma_{\theta R}}{\partial\theta}+\cot\theta\frac{\sigma_{\theta R}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi R}}{\partial\phi}-\frac{1}{R}\left(\sigma_{\theta\theta}+\sigma_{\phi\phi}\right) \\ \frac{\partial\sigma_{R\theta}}{\partial R}+2\frac{\sigma_{R\theta}}{R}+\frac{1}{R}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\cot\theta\frac{\sigma_{\theta\theta}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi\theta}}{\partial\phi}+\frac{\sigma_{\theta R}}{R}-\cot\theta\frac{\sigma_{\phi\phi}}{R} \\ \frac{\partial\sigma_{R\phi}}{\partial R}+2\frac{\sigma_{R\phi}}{R}+\frac{\sin\theta}{R}\frac{\partial\sigma_{\theta\phi}}{\partial\theta}+\cos\theta\frac{\sigma_{\theta\phi}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\frac{1}{R}\left(\sigma_{\phi R}+\sigma_{\phi\theta}\right) \end{bmatrix}+\begin{bmatrix}\rho b_R\\ \rho b_\theta\\ \rho b_\phi\end{bmatrix}=\begin{bmatrix}\rho\frac{dv_R}{dt}\\ \rho\frac{dv_\theta}{dt}\\ \rho\frac{dv_\phi}{dt}\end{bmatrix}$$

Cylindrical-polar coordinates

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix}$$





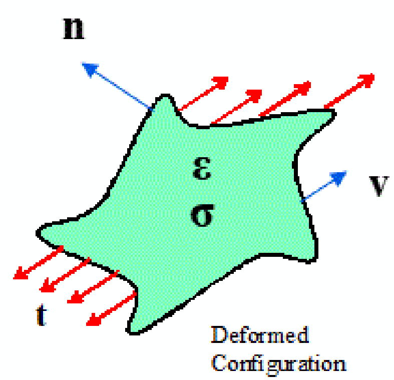
$$\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt}\equiv\begin{bmatrix} \frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta}+\frac{\partial\sigma_{zR}}{\partial z}-\frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{\sigma_{r\theta}}{r}+\frac{\sigma_{\theta r}}{r}+\frac{\partial\sigma_{z\theta}}{\partial z} \\ \frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{rz}}{\partial r}+\frac{\sigma_{rz}}{r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} \end{bmatrix}+\begin{bmatrix}\rho b_r\\ \rho b_\theta\\ \rho bz\end{bmatrix}=\begin{bmatrix}\rho\frac{dv_r}{dt}\\ \rho\frac{dv_\theta}{dt}\\ \rho\frac{dv_z}{dt}\end{bmatrix}$$

6.4 Power and work done by stresses

Rate of work done by stress per unit volume

$$\boxed{\frac{P}{V} = \sigma : \frac{d\varepsilon}{dt} = \sigma_{ij}\frac{d\varepsilon_{ij}}{dt} = \sigma_{11}\frac{d\varepsilon_{11}}{dt} + \sigma_{12}\frac{d\varepsilon_{12}}{dt} + \sigma_{13}\frac{d\varepsilon_{13}}{dt} + \text{etc}}$$

Total work done per unit vol $\quad \dfrac{W}{V} = \displaystyle\int_0^{\varepsilon_{kl}} \sigma_{ij}(\varepsilon_{kl})\, d\varepsilon_{ij}$



Derivation: Consider solid under uniform stress & strain

Rate of work done by ext forces $\quad (P = \underline{F} \cdot \underline{v})$

$$P = \int_A \underline{t} \cdot \underline{v}\, dA$$

Recall $\underline{t} = \underline{n}\,\sigma$

Position on boundary

For uniform strain $\underline{u} = \varepsilon\,\underline{x} \Rightarrow \underline{v} = \frac{d\varepsilon}{dt}\,\underline{x}$

$$\Rightarrow \quad P = \int_A n_i\,\sigma_{ij}\,\frac{d\varepsilon_{jk}}{dt}\,x_k\,dA = \sigma_{ij}\,\frac{d\varepsilon_{jk}}{dt}\int_A n_i\,x_k\,dA$$

Divergence Thm: $\displaystyle\int_A n_i\,x_k\,dA = \int_V \underbrace{\frac{\partial}{\partial x_i}\,x_k}_{=\,\delta_{ik}}\,dV = \delta_{ik}\,V$

$$P = \sigma_{ij}\,\frac{d\varepsilon_{jk}}{dt}\,\delta_{ik} = \sigma_{ij}\,\frac{d\varepsilon_{ji}}{dt} = \sigma_{ij}\,\frac{d\varepsilon_{ij}}{dt}$$

$\sigma_{ij} = \sigma_{ji}$

7 Stress-strain-temperature relations for small strain isotropic elastic materials ("linear elastic")

Background: We have (1) Strain-displacement $\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T)$

(2) EOM $\nabla \cdot \sigma + \rho b = \rho\, \partial v / \partial t$

(3) Boundary Conditions $t = n\sigma$

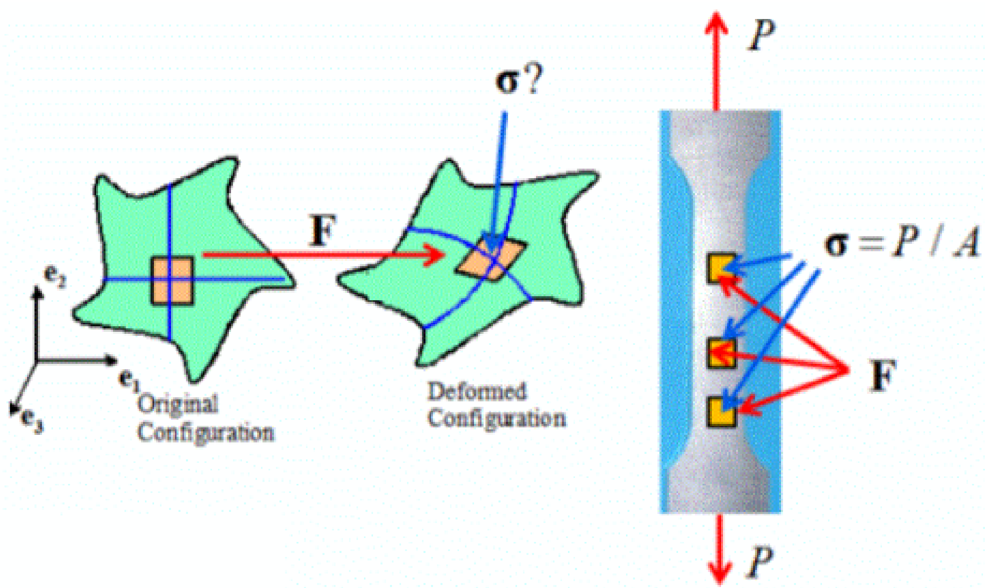
Can't solve! Incomplete! Need to relate σ to ε

Engineering approach

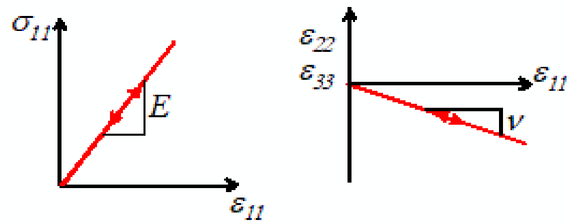
(1) Subject specimen to uniform strain $\Rightarrow \sigma$ uniform

(2) Measure force P, deduce σ

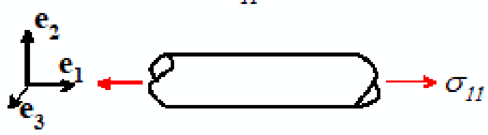
(3) Fit curve to results
- try to use physics



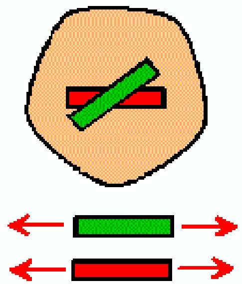
7.1 Stress-strain-temp relations



Experimental Observations



(1) Stress is proportional to strain
(2) Behavior is reversible
(3) For most materials behavior is independent of orientation of specimen wrt material "isotropic"
(4) Increasing temperature produces uniform volumetric strain proportional to temperature



Stress-strain response equal

Fit with curve

Matrix form of $\sigma - \varepsilon$ relation (as in ABAQUS)

$$\underbrace{\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix}}_{\underline{\varepsilon}} = \underbrace{\frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ & & & 0 & 2(1+\nu) & 0 \\ & & & 0 & 0 & 2(1+\nu) \end{bmatrix}}_{[S]\ :\ \text{“Compliance Tensor”}} \underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix}}_{\underline{\sigma}} + \Delta T \alpha \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\underline{\alpha}\ \text{Thermal exp. tensor}}$$

E : Youngs Modulus ν : Poissons ratio

α : thermal expansion coeft

$$\underline{\varepsilon} = [S]\,\underline{\sigma} + \underline{\alpha}\,\Delta T$$

$$\text{Inverse:} \quad \underline{\sigma} = \underbrace{[S]^{-1}}_{[C]\ :\ \text{Stiffness tensor}} \big(\overbrace{\underline{\varepsilon} - \underline{\alpha}\,\Delta T}^{\underline{\varepsilon}^{e}\ :\ \text{“elastic” strain}}\big)$$

Index Notation

$$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} + \alpha\Delta T\,\delta_{ij}$$

$$\sigma_{ij} = C_{ijkl}\left(\varepsilon_{kl} - \alpha\Delta T\,\delta_{ij}\right)$$

or $$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\,\delta_{ij}$$

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$

Example: Uniaxial Tension $\sigma_{11} = \sigma_0$

All others zero

σ - ε relations predict $\varepsilon_{11} = \frac{\sigma_0}{E}$; $\varepsilon_{33} = \varepsilon_{22} = -\frac{\nu}{E}\sigma_0$

All others zero

Other elastic constants

Bulk modulus $K = \dfrac{E}{3(1-2\nu)}$

Significance $\sigma_h = K \dfrac{\Delta V}{V}$ $\quad \sigma_h = \dfrac{1}{3}\,\mathrm{trace}(\sigma)$

$\dfrac{\Delta V}{V} = \mathrm{trace}(\varepsilon)$

Shear modulus $\mu = \dfrac{E}{2(1+\nu)}$

Significance $\sigma_{12} = \mu\, 2\varepsilon_{12}$ $\quad \sigma_{13} = \mu\, 2\varepsilon_{13}$ etc

Lamé modulus $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$

Significance

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} \rightarrow (3\lambda+2\mu)\alpha\Delta\,\delta_{ij}$$

Relations between elastic constants

	LAME MODULUS λ	SHEAR MODULUS μ	YOUNG'S MODULUS E	POISSON'S RATIO ν	BULK MODULUS K
λ,μ			$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ,E		Irrational		Irrational	Irrational
λ,ν		$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$		$\frac{\lambda(1+\nu)}{3\nu}$
λ,K		$\frac{3(K-\lambda)}{2}$	$\frac{9K(K-\lambda)}{3K-\lambda}$	$\frac{\lambda}{3K-\lambda}$	
μ,E	$\frac{\mu(2\mu-E)}{E-3\mu}$			$\frac{E-2\mu}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ,ν	$\frac{2\mu\nu}{1-2\nu}$		$2\mu(1+\nu)$		$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ,K	$\frac{3K-2\mu}{3}$		$\frac{9K\mu}{3K+\mu}$	$\frac{3K-2\mu}{2(3K+\mu)}$	
E,ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$			$\frac{E}{3(1-2\nu)}$
E,K	$\frac{3K(3K-E)}{9K-E}$	$\frac{3EK}{9K-E}$		$\frac{3K-E}{6K}$	
ν,K	$\frac{3K\nu}{(1+\nu)}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$3K(1-2\nu)$		

7.2 Strain energy density

Idea: energy stored in elastic deformation

$U = \frac{1}{2} k \delta^2$

Energy per unit volume

$$U = \frac{1}{2} \underline{\sigma} \cdot \underline{\varepsilon}^e = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}^e = \frac{1}{2}\left(\sigma_{11} \varepsilon_{11}^e + \sigma_{12} \varepsilon_{12}^e + \sigma_{13} \varepsilon_{13}^e + \sigma_{21} \varepsilon_{21}^e + \text{etc} \right)$$

Here $\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}\ \sigma_{13}\ \sigma_{23}]$

$$\underline{\varepsilon}^e = [\varepsilon_{11} - \alpha \Delta T, \varepsilon_{22} - \alpha \Delta T, \varepsilon_{33} - \alpha \Delta T, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]$$

Also useful $U = \frac{1}{2}\left([C]\underline{\varepsilon}^e\right) \cdot \varepsilon^e = \frac{1}{2} C_{ijkl} \varepsilon_{ij}^e \varepsilon_{kl}^e$

$$= \frac{1}{2}\left([S]\underline{\sigma}\right) \cdot \underline{\sigma} = \frac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl}$$

Derivation : $U = \frac{W}{V} = \int_0^{\varepsilon} \sigma_{ij}(\varepsilon^e)\, d\varepsilon_{ij}^e$

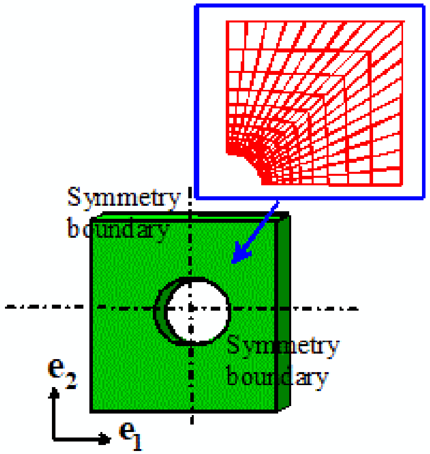
$$= \int_0^{\varepsilon_{ij}^e} C_{ijkl}\, \varepsilon_{kl}^e\, d\varepsilon_{ij}^e$$

$$= \frac{1}{2} C_{ijkl}\, \varepsilon_{kl}^e\, \varepsilon_{ij}^e$$

U will be needed to calculate potential energy of a solid to derive FEA equations

7.3 Simplified stress & strain states

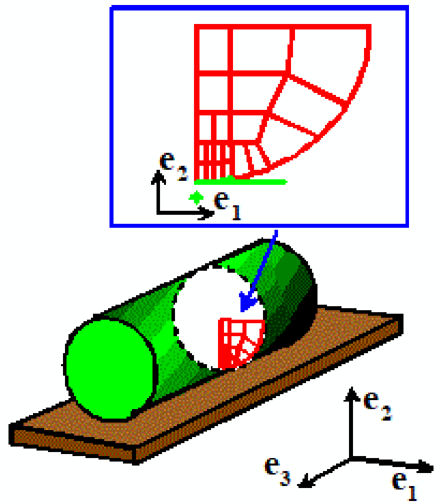
Plane stress : thin sheet loaded in $\{e_1\ e_2\}$ plane



Traction free surfaces $[0\ 0\ 1]\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \\ & \text{etc} & \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\Rightarrow \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$$

Plane Strain Long cylinder with no axial defomation



$u_3 = 0$; u_1, u_2 depend only on x_1, x_2

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$$

Useful elasticity formulas

Matrix form for stress-strain law (3D)

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

For plane strain $\varepsilon_{33}=\varepsilon_{23}=\varepsilon_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{(1+\nu)}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}+(1+\nu)\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\sigma_{33}=\frac{E\nu(\varepsilon_{11}+\varepsilon_{22})}{(1-2\nu)(1+\nu)}+\frac{E\alpha\Delta T}{1-2\nu},\ \sigma_{13}=\sigma_{23}=0$$

For plane stress $\sigma_{33}=\sigma_{23}=\sigma_{13}=0$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{(1-\nu)}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\varepsilon_{33}=-\frac{\nu}{E}(\sigma_{11}+\sigma_{22})+\alpha\Delta T$$