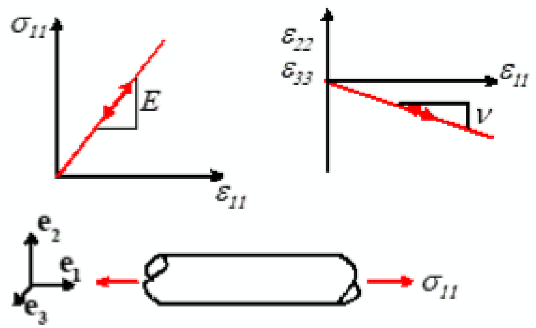
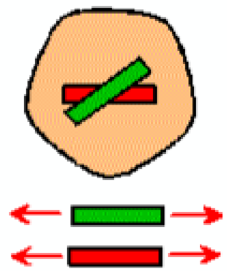
Review



Stress-Strain-Temperature relations for elastic solids

Assumptions:

1. Small deformations
2. Isotropic material
3. Strain linearly related to stress / temp
4. Perfectly reversible



Stress-strain response equal

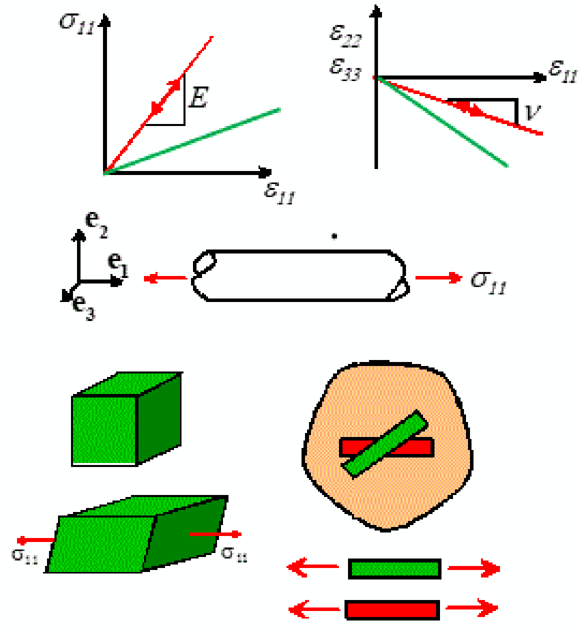
Matrix form for stress-strain law (3D)

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\Delta T\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\nu)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}$$

7.4 Anisotropic Materials

Anisotropy means
(1) Response depends on orientation of specimen wrt to material
(2) Tensile stress may induce shear strain



Stress-strain response differs

Options in ABAQUS:
(1) General
(2) Othotropic

Examples: composites ; wood ; single xtals

General anisotropic $\sigma - \varepsilon$ relations

$$\underline{\sigma} = [C]\,(\underline{\varepsilon} - \underline{\alpha}\,\Delta T)$$

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]$$

$$\underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]$$

Now $[C]$ is a nonzero symmetric 6×6 matrix

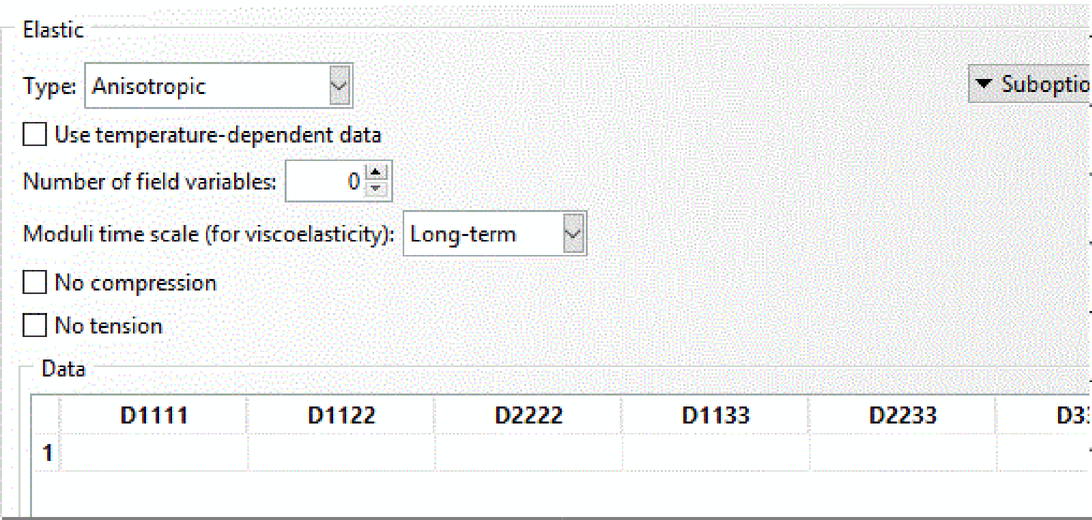
$$[C] = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & D_{1112} & \text{etc} \\ & D_{2222} & & & \\ \text{Sym} & & & & \end{bmatrix}$$

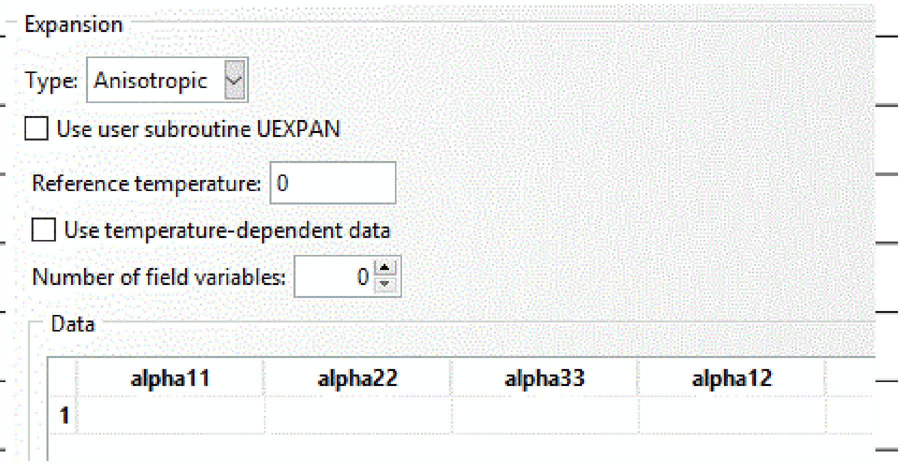
D_{ijkl} is the same as C_{ijkl} (4th order tensor)

Some books use $D_{1111} = C_{11}$ $D_{1122} = C_{12}$ etc

$$\underline{\alpha} = [\alpha_{11}\ \alpha_{22}\ \alpha_{33},\ 2\alpha_{12}\ 2\alpha_{13}\ 2\alpha_{23}]$$

2 may or may not be used





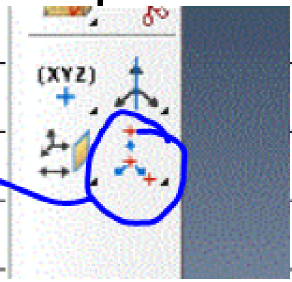
ABAQUS menus : enter numbers (21 elastic consts!)

Note values of D_{ijkl} depend on basis for σ, ε

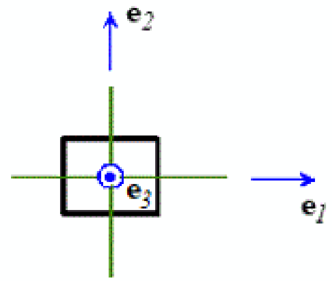
In ABAQUS you can specify the orientation of material axes (basis for σ-ε law)

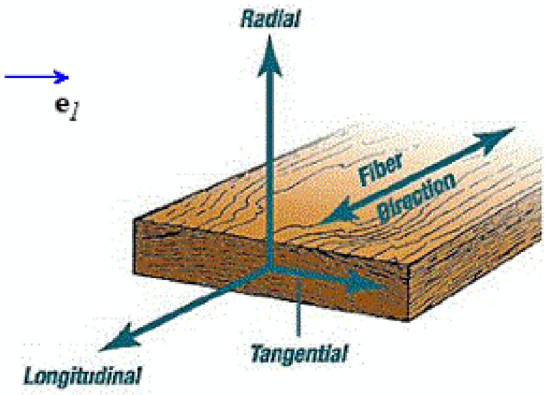
(1) Create a local coordinate system

(2) Assign material orientation to part



Orthotropic materials





Orthotropic materials have 3 symmetry planes

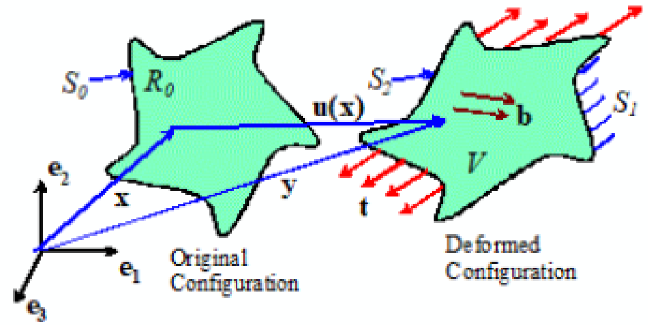
If we choose $\{e_1, e_2, e_3\}$ to be perpendicular to symmetry planes then $[C]$ has form

$$[C] = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & 0 & 0 & 0 \\ & D_{2222} & D_{2233} & 0 & 0 & 0 \\ & & D_{3333} & 0 & 0 & 0 \\ & & & D_{1212} & 0 & 0 \\ & & & 0 & D_{1313} & 0 \\ & & & 0 & 0 & D_{2323} \end{bmatrix}$$

also $\underset{\sim}{\alpha} = [\alpha_{11} \ \alpha_{22} \ \alpha_{33} \ 0 \ 0 \ 0]$

Will need to specify orientation wrt part

8) Solving static boundary value problems for linear elastic solids



Assume: (1) Small defomations
(2) Elastic, isotropic material

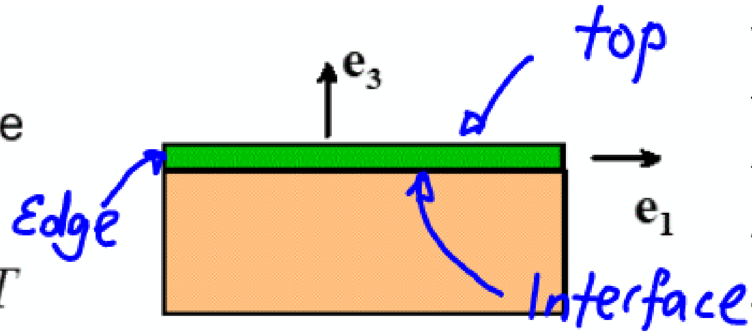
Given: Body force $\underline{b}$
Temperature ΔT
Boundary loading : either $\underline{u} = \underline{u}^*$ given on S_1
$\underline{t}$ given on S_2
[or mixed $\underline{t}$, $\underline{u}$ components]

Find: $[\underline{u}, \varepsilon, \sigma]$ satisfying

(1) Stain-displacement $\quad \varepsilon = (\nabla \underline{u} + (\nabla \underline{u})^T)/2$
(2) Material law $\quad \sigma = C(\varepsilon - \alpha \Delta T)$
(3) Static equilibrium $\quad \nabla \cdot \sigma + \rho \underline{b} = \underline{0}$ (statics)
(4) Boundary conditions $\quad \underline{u} = \underline{u}^*$ on S_1 $\quad \underline{n}\,\sigma = \underline{t}$ on S_2

8.1 Solving simple problems (approximately) using physical reasoning

Example: Film, elastic constants (E, ν, α) on a large rigid substrate with $\alpha = 0$



Initially stress free, then heated to temp ΔT

Find stress in film and strain energy density

(Focus on behavior away from edges)

Observations:
(1) Substrate is rigid $\Rightarrow$ $\underline{u} = \underline{0}$ in substrate
(2) Top and edge are traction free
(3) Film remains bonded $\Rightarrow$ film & substrate have same $\underline{u}$ at interface

Boundary Conditions

(a) Top $\underline{n}\,\sigma = \underline{0} \Rightarrow [0\ 0\ 1]\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ & & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \sigma_{33} = 0 \quad \sigma_{13} = 0 \quad \sigma_{23} = 0$ (1)

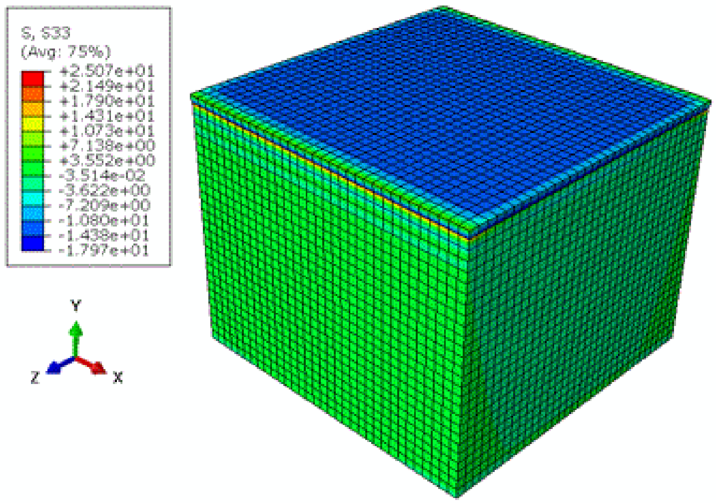
Edges $\underline{n} = [1\ 0\ 0]$ eg $\Rightarrow \sigma_{11} = \sigma_{12} = \sigma_{13} = 0$

(b) Interface $u_1(x_1, x_2, 0) = u_2(x_1, x_2, 0)$
$= u_3(x_1, x_2, 0) = 0$

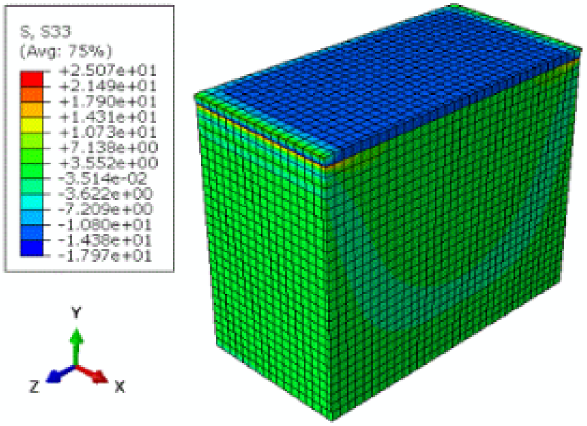
Hence $\varepsilon_{11} = \partial u_1/\partial x_1 = 0 \quad \varepsilon_{22} = \partial u_2/\partial x_2 = 0$
$\varepsilon_{12} = (\partial u_1/\partial x_2 + \partial u_2/\partial x_1)/2 = 0$

Condition (1) $\Rightarrow$ plane stress state @ surface

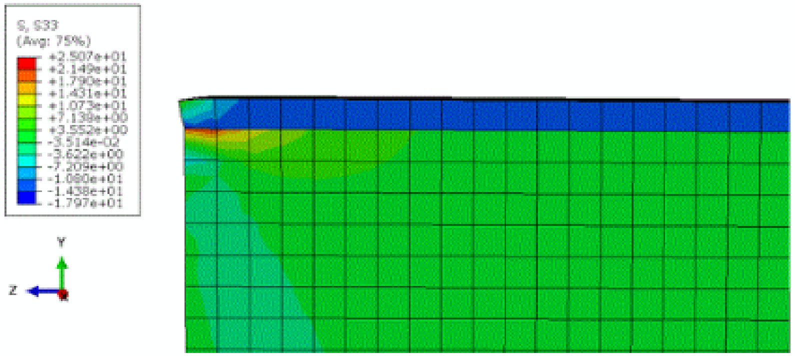
FEA solution



Away from edges stress is uniform



Cut through center



About 3 × film thicknesses away from edge σ is uniform

$$\boxed{\text{Assume } [\sigma, \varepsilon] \text{ are constant}}$$

$\Rightarrow \nabla \cdot \underline{\sigma} + \underline{b} = \underline{0}$ automatically $\nabla \cdot \sigma = 0$!

$\boxed{\text{Satisfy BCs}}$: we must have plane stress

σ-ε relations for plane stress

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{(1-\nu^2)}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}\underbrace{\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}}_{=0} - \frac{E\alpha\Delta T}{(1-\nu)}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11}+\sigma_{22}) + \alpha\Delta T$$

$\Rightarrow$

$$\boxed{\begin{gathered}\sigma_{11} = \sigma_{22} = -\frac{E\alpha\Delta T}{1-\nu}\\ \text{All other } \sigma_{ij} = 0\\ \varepsilon_{33} = \frac{(1+\nu)}{(1-\nu)}\alpha\Delta T\\ \text{all other } \varepsilon_{ij} = 0\end{gathered}}$$

Strain energy density $U = \frac{1}{2}\sigma_{ij}\varepsilon^e_{ij}$

$$\varepsilon^e_{ij} = \varepsilon_{ij} - \alpha\,\delta_{ij}\,\Delta T \Rightarrow \varepsilon^e_{11} = \varepsilon^e_{22} = -\alpha\Delta T$$

$$\boxed{U = \frac{1}{2}\left\{\sigma_{11}\varepsilon^e_{11} + \sigma_{22}\varepsilon^e_{22}\right\} = \frac{E}{(1-\nu)}(\alpha\Delta T)^2}$$

Typical values $E \sim 100$ GPa

$\alpha \sim 10^{-5}\ K^{-1}$

$\nu \sim 0.3$

$\Delta T \sim 100\,^{\circ}C$

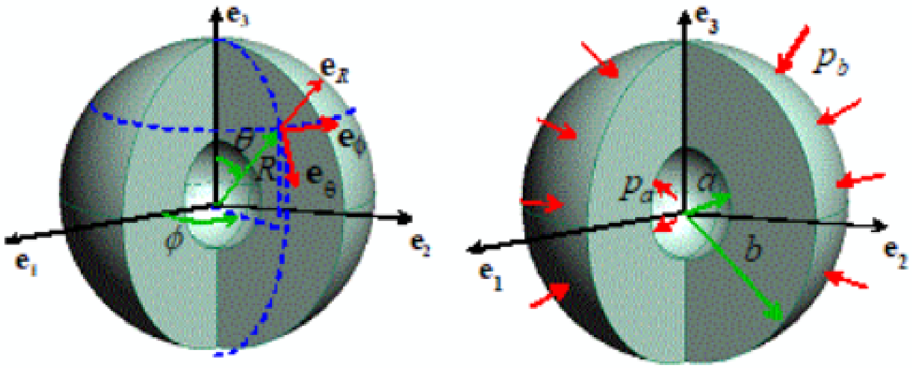
$\sigma_{11} \sim 100$ MPa quite large!

$U \sim 0.1\ MJ/m^3$

for comparison energy in specific heat capacity 400 MJ/m^3

8.2 Spherically symmetric solutions

Consider spherical shell $a < R < b$



Given : Radial body force $b(R)\,\underline{e}_R$
Temp $\Delta T(R)$

Either pressure p_a, p_b on $R = a, b$

or radial displacement $u_R(a) = u_a^*$ $u_R(b) = u_b^*$

Assume radial displacement field $\underline{u} = u_R(R)\,\underline{e}_R$

Solve for $[u_R(R), \varepsilon, \sigma]$