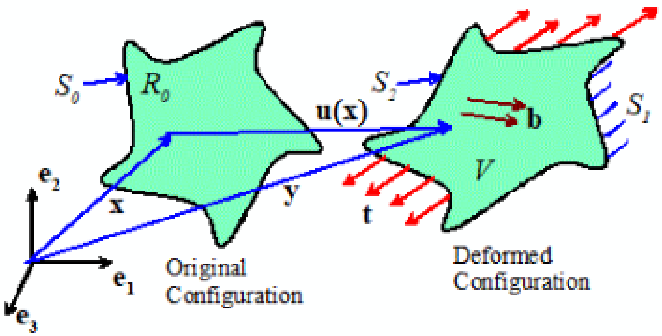
Review

Static boundary value problems for linear elastic solids



Assumptions:

1. Small displacements
2. Isotropic, linear elastic material

Given:

1. Traction or displacement on all exterior surfaces
2. Body force and temperature distribution

Find: $[u_i, \varepsilon_{ij}, \sigma_{ij}]$

Governing Equations:

1. Strain-displacement relation (you can use the compatibility equation instead)

$$\varepsilon_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i) / 2 \qquad \boldsymbol{\varepsilon} = \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] / 2$$

2. Stress-strain law

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} \right) - \frac{E\alpha\Delta T}{(1-2\nu)}\delta_{ij} \qquad \boldsymbol{\sigma} = \frac{E}{1+\nu}\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu} trace(\boldsymbol{\varepsilon})\mathbf{I} \right) - \frac{E\alpha\Delta T}{(1-2\nu)}\mathbf{I}$$

3. Equilibrium $\dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = 0 \qquad \nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0}$

4. Boundary conditions on external surfaces
   1. Where displacements are prescribed $u_i = u_i^* \quad \mathbf{u} = \mathbf{u}^*$
   2. Where tractions are prescribed $n_j \sigma_{ji} = t_i \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{t}$

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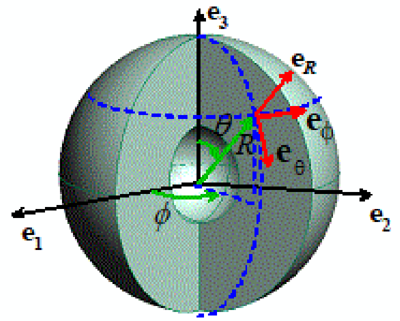


Solving problems with spherical symmetry

Assumptions:

1. Solid is a spherical shell $a < R < b$
2. Isotropic, linear elastic material
3. Body force is radial $\mathbf{b} = b(R)\mathbf{e}_R$
4. Temperature varies only in radial direction $\Delta T(R)$

Boundary conditions:

1. Either given (radial) pressure or radial displacement u_a^* on $R = a$
2. Either given pressure p_b or radial displacement u_b^* on $R = b$

Observation: spherical symmetry suggests points in sphere will move only radially $\Rightarrow \mathbf{u} = u(R)\mathbf{e}_R$

Goal: Simplify elasticity equations for this displacement and solve for $u(R), \boldsymbol{\varepsilon}, \boldsymbol{\sigma}$

Simplified governing equations

Strain - Displacement $\varepsilon = [\nabla \underline{u} + (\nabla \underline{u})^T]/2$

Here $u_R = u(R)$ $u_\theta = u_\phi = 0$

$$\begin{bmatrix} \varepsilon_{RR} & \varepsilon_{R\theta} & \varepsilon_{R\phi} \\ & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\ & & \varepsilon_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \partial u/\partial R & 0 & 0 \\ & u/R & 0 \\ & & u/R \end{bmatrix}$$

$$\nabla \mathbf{v} \equiv \begin{bmatrix} \dfrac{\partial v_R}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_R}{\partial \theta} - \dfrac{v_\theta}{R} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_R}{\partial \phi} - \dfrac{v_\phi}{R} \\ \dfrac{\partial v_\theta}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_R}{R} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_\theta}{\partial \phi} - \cot\theta\dfrac{v_\phi}{R} \\ \dfrac{\partial v_\phi}{\partial R} & \dfrac{1}{R}\dfrac{\partial v_\phi}{\partial \theta} & \dfrac{1}{R\sin\theta}\dfrac{\partial v_\phi}{\partial \phi} + \cot\theta\dfrac{v_\theta}{R} + \dfrac{v_R}{R} \end{bmatrix}$$

$$\varepsilon_{RR} = \partial u/\partial R \qquad \varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = u/R \qquad (1)$$

Stress - strain relation (Same as Cartesian with $\varepsilon_{11} = \varepsilon_{RR}$ etc)

$$\sigma = \begin{bmatrix} \sigma_{RR} & \sigma_{R\theta} & \sigma_{R\phi} \\ & \sigma_{\theta\theta} & \sigma_{\phi\theta} \\ & & \sigma_{\phi\phi} \end{bmatrix}$$

Here $\sigma_{R\theta} = \sigma_{R\phi} = \sigma_{\phi\theta} = 0$ $\sigma_{\theta\theta} = \sigma_{\phi\phi}$

$$\begin{bmatrix} \sigma_{RR} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{RR} \\ \varepsilon_{\theta\theta} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (2)$$

Equilibrium

$$\nabla \cdot \underline{\sigma} + \rho \underline{b} = \underline{0}$$

$$\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt}\equiv\begin{bmatrix} \frac{\partial\sigma_{RR}}{\partial R}+2\frac{\sigma_{RR}}{R}+\frac{1}{R}\frac{\partial\sigma_{\theta R}}{\partial\theta}+\cot\theta\frac{\sigma_{\theta R}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi R}}{\partial\phi}-\frac{1}{R}\left(\sigma_{\theta\theta}+\sigma_{\phi\phi}\right) \\ \frac{\partial\sigma_{R\theta}}{\partial R}+2\frac{\sigma_{R\theta}}{R}+\frac{1}{R}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+\cot\theta\frac{\sigma_{\theta\theta}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi\theta}}{\partial\phi}+\frac{\sigma_{\theta R}}{R}-\cot\theta\frac{\sigma_{\phi\phi}}{R} \\ \frac{\partial\sigma_{R\phi}}{\partial R}+2\frac{\sigma_{R\phi}}{R}+\frac{\sin\theta}{R}\frac{\partial\sigma_{\theta\phi}}{\partial\theta}+\cos\theta\frac{\sigma_{\theta\phi}}{R}+\frac{1}{R\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi}+\frac{1}{R}\left(\sigma_{\phi R}+\sigma_{\phi\theta}\right)\end{bmatrix}+\begin{bmatrix}\rho b_R\\ \rho b_\theta\\ \rho b_\phi\end{bmatrix}=\begin{bmatrix}\rho\frac{dv_R}{dt}\\ \rho\frac{dv_\theta}{dt}\\ \rho\frac{dv_\phi}{dt}\end{bmatrix}$$

← Non-trivial

← $0 = 0$

← $0 = 0$

$$\Rightarrow \frac{\partial \sigma_{RR}}{\partial R} + \frac{2}{R}\left(\sigma_{RR} - \sigma_{\theta\theta}\right) + \rho\, b(R) = 0 \qquad (3)$$

Boundary conditions

on $R = a$ either $u(R) = u_a^*$ or $\underline{n}\,\sigma = p_a \underline{e}_R$

$\underline{n} = -\underline{e}_R$

$$\Rightarrow [-1 \quad 0 \quad 0]\begin{bmatrix} \sigma_{RR} & 0 & 0 \\ & \sigma_{\theta\theta} & 0 \\ & & \sigma_{\theta\theta} \end{bmatrix} = \begin{bmatrix} p_a \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sigma_{RR} = -p_a$$

on $R = b$ either $u(R) = u_b^*$ or $\sigma_{RR} = -p_b$

Solution: Subst (1) into (2) & then into (3) (MATLAB)

```
syms EE nu rho alpha C1 C2 R eRR eqq sRR sqq real
syms dT(R) u(R) b(R)
assume(EE>0);
C1 = EE/(1+nu)/(1-2*nu);
C2 = EE*alpha*dT(R)/(1-2*nu);
eRR = diff(u(R),R); eqq = u(R)/R;
sRR = C1*((1-nu)*eRR + 2*nu*eqq) - C2;
sqq = C1*(nu*eRR+eqq) - C2;
equil = simplify(diff(sRR,R) + 2*(sRR-sqq)/R);
simpler = simplify(subs(equil,dT(R),0)/C1/(1-nu)) +...
          simplify(subs(equil,u(R),0)/C1/(1-nu)) - rho*b(R)/C1/(1-nu)==0
```

simpler =

$$\frac{2\,R\,\frac{\partial}{\partial R}\,u(R) - 2\,u(R) + R^2\,\frac{\partial^2}{\partial R^2}\,u(R)}{R^2} + \frac{\alpha\,(\nu+1)\,\frac{\partial}{\partial R}\,\mathrm{dT}(R)}{\nu-1} - \frac{\rho\,b(R)\,(2\,\nu-1)\,(\nu+1)}{\mathrm{EE}\,(\nu-1)} = 0$$

Hence

$$\underbrace{\frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2u}{R^2}}_{\frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}(R^2u)\right\}} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{d\Delta T}{dR} + \frac{(1-2\nu)(1+\nu)\,b(R)}{(1-\nu)E}$$

Can solve for any ΔT, b by integration

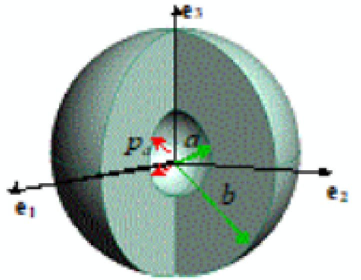
- Solve for integration constants using BCs

Example: Sphere, no body force, no temp change, subjected to pressure

p_a on $R=a$ traction free on $R=b$

Find stress in the sphere

If the sphere has yield stress Y find the pressure that causes yield.



Integrate

$$\frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}(R^2 u)\right\} = 0 \Rightarrow \frac{1}{R^2}\frac{d}{dR}(R^2 u) = C$$

$$\Rightarrow u = \frac{1}{3}CR + \frac{D}{R^2}$$

C, D are integration constants

Find C, D by: (1) Find $\varepsilon_{RR}, \varepsilon_{\theta\theta}$

(2) Find σ_{RR}

(3) $\left.\begin{array}{ll}\sigma_{RR} = -p_a & R=a \\ \sigma_{RR} = 0 & R=b\end{array}\right\}$ 2 eqs

Use MATLAB

```
syms EE nu a b pa R C D  real
C1 = EE/(1+nu)/(1-2*nu);
u = C*R/3 + D/R^2;
eRR = diff(u,R); eqq = u/R;
sRR = C1*((1-nu)*eRR + 2*nu*eqq);
sqq = C1*(nu*eRR+eqq);
BC1 = subs(sRR,R,a) == -pa; BC2 = subs(sRR,R,b)==0;
[Csol,Dsol] = solve([BC1,BC2],C,D);
u = simplify(subs(u,[C,D],[Csol,Dsol]))
sRR = simplify(subs(sRR,[C,D],[Csol,Dsol]))
sqq = simplify(subs(sqq,[C,D],[Csol,Dsol]))
```

u =

$$-\frac{a^3\,\mathrm{pa}\,(b^3\,\nu - 4\,R^3\,\nu + 2\,R^3 + b^3)}{2\,\mathrm{EE}\,R^2\,(a^3 - b^3)}$$

sRR =

$$-\frac{a^3\,\mathrm{pa}\,(R^3 - b^3)}{R^3\,(a^3 - b^3)}$$

sqq =

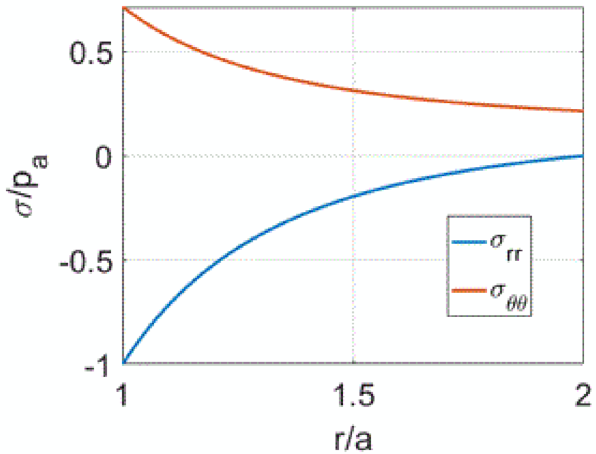
$$-\frac{a^3\,\mathrm{pa}\,(2\,R^3 + b^3)}{2\,R^3\,(a^3 - b^3)}$$

Hence $$\boxed{\sigma_{RR} = -p_a \frac{a^3}{R^3} \frac{(b^3 - R^3)}{(b^3 - a^3)} \qquad \sigma_{\theta\theta} = p_a \frac{a^3}{2R^3} \frac{(b^3 + 2R^3)}{(b^3 - a^3)}}$$

σ_{RR} : Compressive

$\sigma_{\theta\theta}$: Tensile

all others zero



Pressure to cause yield

Yield Criterion : $\sigma_e = \sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2]} = Y$

Here $\sigma_1 = \sigma_{\theta\theta}$ $\sigma_2 = \sigma_{\theta\theta}$ $\sigma_3 = \sigma_{RR}$ (principal stresses)

$$\Rightarrow \sigma_e = |\sigma_{\theta\theta} - \sigma_{RR}| = \frac{3}{2} p_a \frac{a^3 b^3}{R^3 (b^3-a^3)}$$

Note σ_e varies with R

Yield occurs if $\sigma_e > Y$ anywhere $\Rightarrow$ find max σ_e

Max @ $R=a$ (yield first occurs at inner wall)

$$\Rightarrow \boxed{\text{yield pressure} \quad p_a^{yield} = \frac{2}{3} \frac{(b^3-a^3)}{b^3} Y}$$

8.4 Features of solutions to linear elasticity problems

[no contact with unknown contact area]

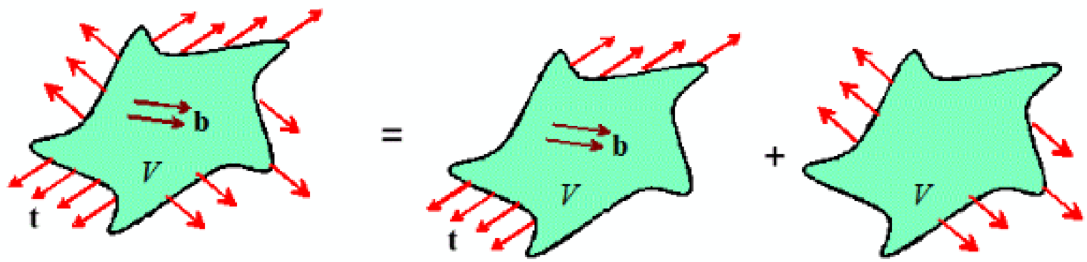
(1) Solution exists & is unique

(2) Solutions are linear: $[\underline{u}, \varepsilon, \sigma]$ are proportional to load

eg for sphere $\sigma_{\theta\theta} = \dfrac{a^3}{2R^3} \dfrac{(b^3+2R^3)}{(b^3-a^3)} p_a$

- double p_a, double $\sigma_{\theta\theta}$

(3) Can superpose solutions



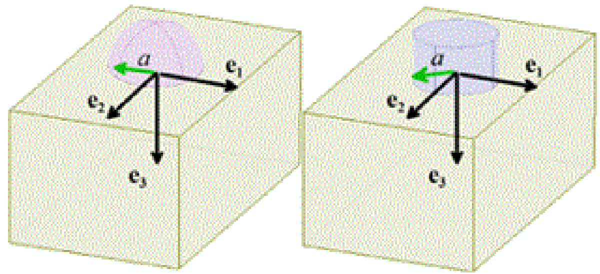
If $\left.\begin{array}{l}[\underline{u}^{(1)}, \varepsilon^{(1)}, \sigma^{(1)}] \ \& \\ [\underline{u}^{(2)}, \varepsilon^{(2)}, \sigma^{(2)}]\end{array}\right\}$ are solutions

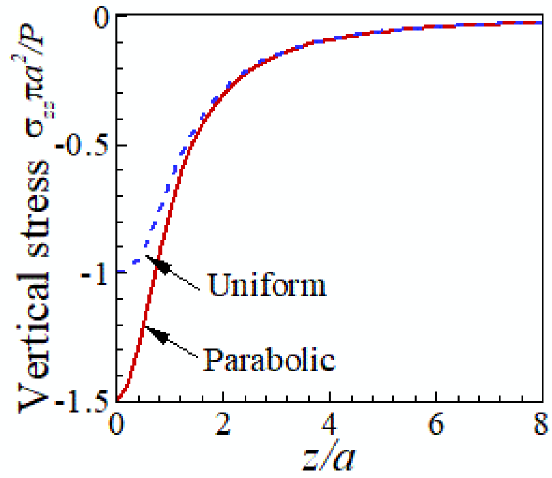
constants

Then $[\alpha \underline{u}^{(1)} + \beta \underline{u}^{(2)}, \alpha \varepsilon^{(1)} + \beta \varepsilon^{(2)}, \alpha \sigma^{(1)} + \beta \sigma^{(2)}]$ is a sol

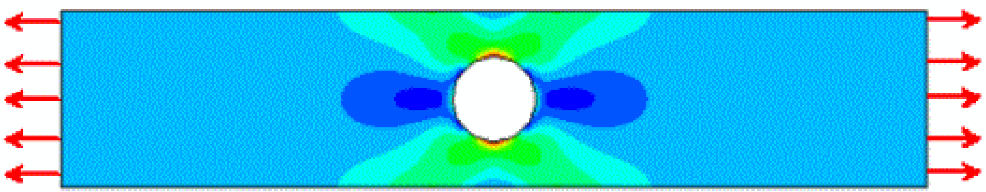
(4) Saint Venant's principle

Version (1): If two pressure distributions acting on a surface exert the same resultant force & moment, they induce the same $[\underline{u}, \varepsilon, \sigma]$ far from the loaded area





Version (2): a local geometric feature in a solid that has traction free surfaces only (usually) influences stresses in a region approx 3 x feature size



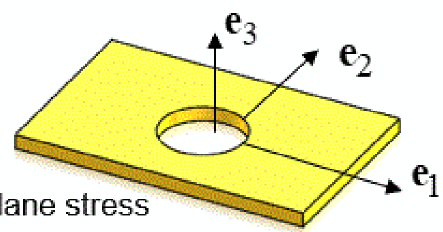
8.5 Solutions to 2D elasticity problems using "Airy function"

- One of several "stress/displacement potential" method

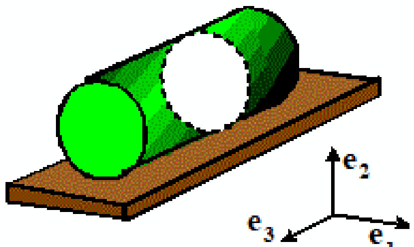
- Basic idea is to replace 6 PDEs for $[\underline{u}, \varepsilon, \sigma]$ with one PDE for a potential & then derive solution for $[\underline{u}, \varepsilon, \sigma]$ from new PDE

Assumptions

(1) 2D (Plane stress or strain)
(2) Assume no body forces $\Delta T = 0$
(3) Isotropic, linear elastic material



Plane stress



Plane strain