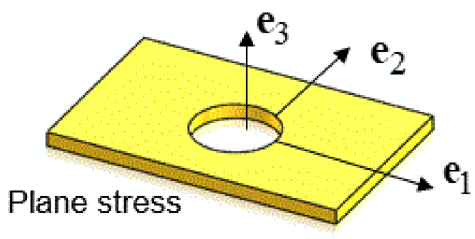
Review

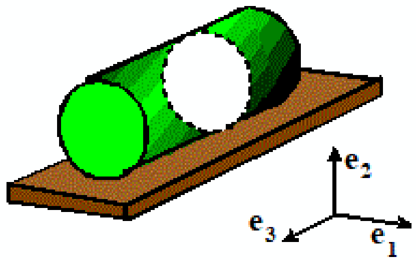
'Airy Function' solution to 2D elasticity problems

Example of a 'potential function' method
Goal is to replace 6 coupled PDEs for unknowns with a simple PDE



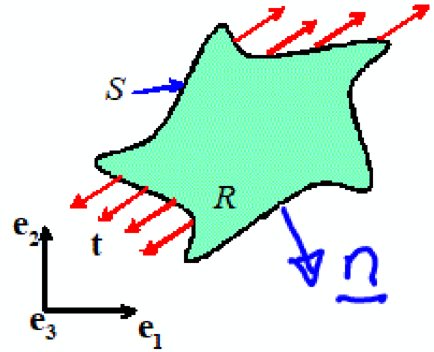
Assumptions:

1. Small displacements
2. Plane solid (plane stress or strain)
3. Isotropic, linear elastic material
4. Neglect body force and temperature



Method works best if tractions are specified on boundary

Airy Function Method



2D Problem : Goal is to find

$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix}$ $\quad \varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{bmatrix}$ $\quad \underline{u} = u_1 \underline{e}_1 + u_2 \underline{e}_2$

$\underline{n} = n_1 \underline{e}_1 + n_2 \underline{e}_2$

Focus on traction BVP $\quad \underline{t} = t_1 \underline{e}_1 + t_2 \underline{e}_2$

Procedure : (1) Find "Airy Function" $\phi(x_1, x_2)$ satisfying

(1) $\nabla^4 \phi = 0 \Rightarrow \dfrac{\partial^4 \phi}{\partial x_1^4} + \dfrac{2\partial^4 \phi}{\partial^2 x_1 \partial^2 x_2} + \dfrac{\partial^4 \phi}{\partial x_2^4} = 0$

(2)
$$\left.\begin{aligned} n_1 \partial^2\phi/\partial x_2^2 - n_2 \partial^2 \phi/\partial x_1 \partial x_2 &= t_1 \\ -n_1 \partial^2\phi/\partial x_1 \partial x_2 + n_2 \partial^2 \phi/\partial x_1^2 &= t_2 \end{aligned}\right\} \text{ on boundary}$$

Then find $[\sigma, \varepsilon, \underline{u}]$ as follows

(a) Stresses :

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

(b) Strains : use plane $\sigma - \varepsilon$ relation (see L8)

Plane strain

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{(1+\nu)}{E} \begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

Plane stress

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

(c) Displacements: Integrate strains (see discussion of compatibility L #5)

Proof of the Airy Representation

Airy Function

$$\nabla^4 \phi \equiv \frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = 0$$

Stress

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} \qquad \sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} \qquad \sigma_{12} = \sigma_{21} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2}$$

$$\sigma_{33} = 0 \qquad \text{(Plane Stress)}$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \qquad \text{(Plane Strain)}$$

$$\sigma_{23} = \sigma_{13} = 0$$

Strain

$$\varepsilon_{11} = \frac{1+\nu}{E}\sigma_{11} - \frac{\nu}{E}(1+\beta\nu)(\sigma_{11}+\sigma_{22})$$

$$\varepsilon_{22} = \frac{1+\nu}{E}\sigma_{22} - \frac{\nu}{E}(1+\beta\nu)(\sigma_{11}+\sigma_{22})$$

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12} \qquad \beta = 1 \ \text{(Plane Strain)}$$

$$\beta = 0 \ \text{(Plane Stress)}$$

Equilibrium

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0 \qquad \frac{\partial}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_2^2}\right) + \frac{\partial}{\partial x_2}\left(-\frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right) = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = 0 \qquad \frac{\partial}{\partial x_1}\left(-\frac{\partial^2 \phi}{\partial x_1 \partial x^2}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial^2 \phi}{\partial x_1^2}\right) = 0$$

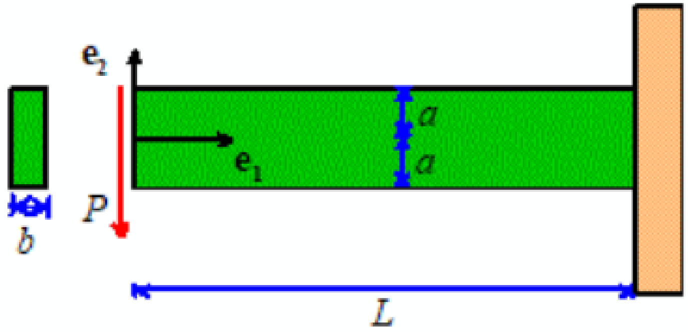
Compatibility

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \qquad \frac{1+\nu}{E}\left(\frac{\partial^2 \sigma_{11}}{\partial x_2^2} + \frac{\partial^2 \sigma_{22}}{\partial x_1^2}\right) - \frac{\nu}{E}(1+\beta\nu)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)(\sigma_{11}+\sigma_{22}) - 2\frac{1+\nu}{E}\frac{\partial^2 \sigma_{12}}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^4 \phi}{\partial x_2^4} + \frac{\partial^4 \phi}{\partial x_1^4} - \frac{\nu(1+\beta\nu)}{1+\nu}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2}\right) + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} = 0$$

$$\frac{\partial^4 \phi}{\partial x_1^4} + 2\frac{\partial^4 \phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \phi}{\partial x_2^4} = 0$$

Example : Cantilever Beam, rectangular x-sect



Boundary Conditions :

(a) Top & Bottom traction free $\underline{t} = 0$

(b) Left hand end use approx boundary condition

$$\int_A \underline{t}\, dA = -P\underline{e}_2 \quad \text{(specify resultant force)}$$

(c) Right hand end enforce $\underline{u} = \underline{0}$ approximately

Consider Airy function

$$\phi = \frac{3P}{4ab} x_1 x_2 \left(\frac{x_2^2}{a^2} - 3 \right)$$

Stresses:

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial x_2^2} = \frac{3P}{4ab} \, 6 \, \frac{x_1 x_2}{a^2}$$

$$\sigma_{22} = \frac{\partial^2 \phi}{\partial x_1^2} = 0$$

$$\sigma_{12} = -\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = -\frac{3P}{4ab} \, 3 \left(\frac{x_2^2}{a^2} - 1 \right)$$

Strains & displacements follow

Check: (a) $\nabla^4 \phi = 0$ (4^{th} derivatives all zero)

(b) Boundary conditions

Top & bottom $\underline{t} = \underline{n}\,\sigma = 0$ on $x_2 = \pm a$

$$\underline{n} = \pm \underline{e}_2$$

$$\pm [0 \quad 1] \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \pm [\sigma_{12} \quad \sigma_{22}] = [0 \quad 0]$$ ✓ exact

Left end : $\int_{A} \underline{t}\, dA = -P\underline{e}_2$? $\quad \underline{t} = \underline{n}\,\sigma \quad \underline{n} = -\underline{e}_1$

$$\underline{n}\,\sigma = [-1 \quad 0] \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = [-\sigma_{11} \quad -\sigma_{12}] \text{ on } x_1 = 0$$

$$\Rightarrow \int_A \underline{t}\, dA = b\int_{-a}^{a} \left\{ 0\,\underline{e}_1 + \frac{3P}{4ab} 3\left(\frac{x_2^2}{a^2} - 1\right)\underline{e}_2 \right\} dx_2 = -P\underline{e}_2$$ ✓ (approx)

Displacements (calculated in L#5)

$$u_1 = Cx_1^2 x_2 - \frac{C}{3}(2+\nu)x_2^3 + 2(1+\nu)Ca^2 x_2 - \omega x_2 + B$$

$$u_2 = -\nu C x_1 x_2^2 - \frac{C}{3}x_1^3 + \omega x_1 + A$$

Impossible to satisfy $u_1 = u_2 = 0$ on $x_1 = L$ for all x_2

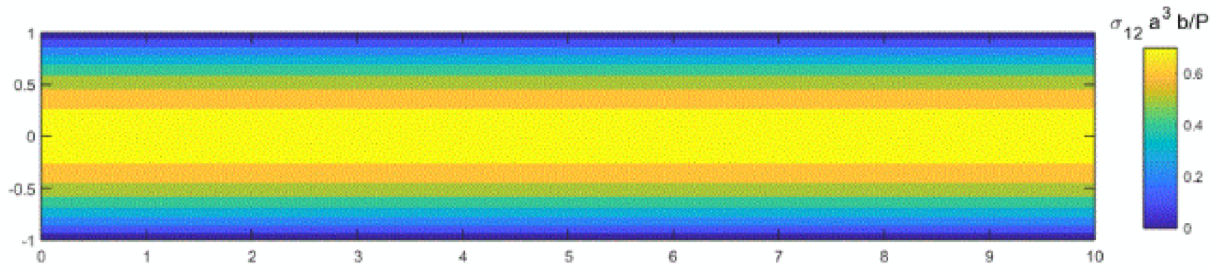
We can get an approx solution by making
$u_1 = u_2 = 0$ on $x_1 = L$ $x_2 = 0$ (solve for A, B)

Can also make $\frac{\partial u_2}{\partial x_1} = 0$ on $x_2 = 0$

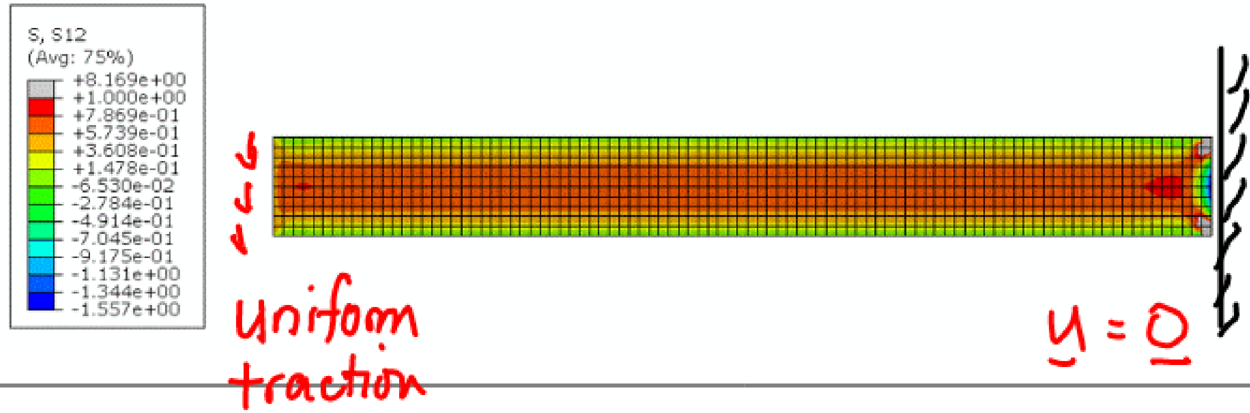
(slope of centerline zero) $\Rightarrow$ find ω

Compare with FEA

Near ends FEA & Airy differ (BCs in airy are not exact)



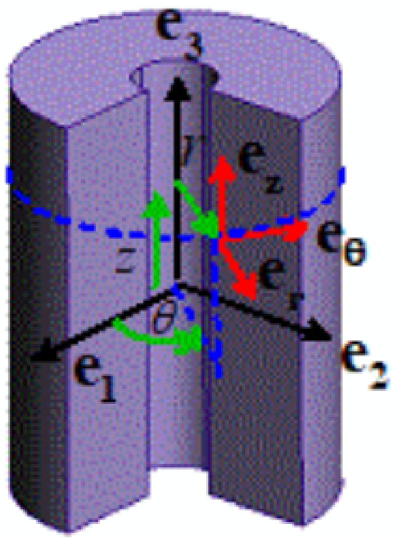
Away from ends FEA & Airy give same shear stress



(Saint Venant says this should happen)

Airy function in cylindrical-polar coords

Goal: Find $\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix}$ $\quad \varepsilon = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} \end{bmatrix}$



$$\underline{u} = u_r \underline{e_r} + u_\theta \underline{e_\theta}$$

Biharmonic equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0$$

Stresses :

$$\sigma_{rr} = \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} \qquad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta} \right)$$

Strain - stress same as before

$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix}=\frac{(1+\nu)}{E}\begin{bmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$$

Plane Strain

$$\begin{bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ 2\varepsilon_{r\theta}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{bmatrix}$$

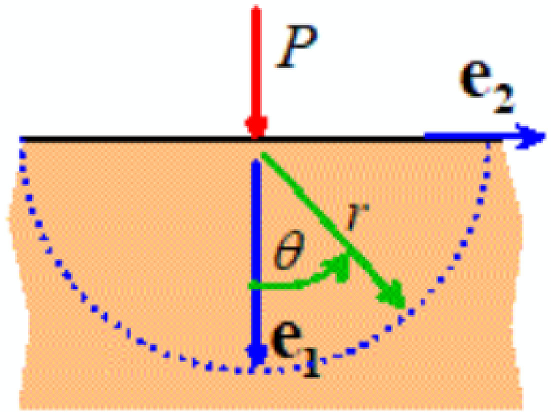
Plane Stress

Strain - displacement

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right\}$$

Example : Line force on surface of large elastic solid



P is force per unit length out of plane

Airy function $\phi = \frac{-P}{\pi} r\theta \sin\theta$

Use formulas (MATLAB)

```
syms phi P theta r srr sqq srq
phi = -P*r*theta*sin(theta)/pi;
laplacian = simplify(diff(phi,r,2) + 1/r*diff(phi,r) + 1/r^2*diff(phi,theta,2));
biharmonic = simplify(diff(laplacian,r,2) + 1/r*diff(laplacian,r) + 1/r^2*diff(laplacian,theta,2))
srr = simplify(diff(phi,r)/r + diff(phi,theta,2)/r^2)
sqq = simplify(diff(phi,r,2))
srq = -simplify(diff(diff(phi,theta)/r,r))
```

biharmonic = 0

srr =

$$-\frac{2P\cos(\theta)}{r\pi}$$

sqq = 0

srq = 0

$$\sigma_{rr} = -\frac{2P}{\pi r}\cos\theta \qquad \sigma_{\theta\theta} = \sigma_{r\theta} = 0$$

Check boundary conditions

(a) On surface $\underline{t} = \underline{0}$ except at origin

$\underline{t} = \underline{n}\,\sigma \qquad \underline{n} = \pm \underline{e}_\theta$ on $\theta = \pm \pi/2$

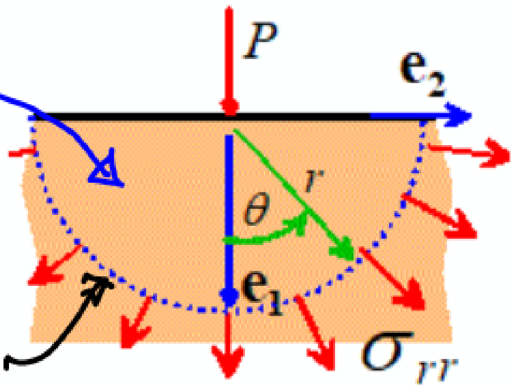
$$\Rightarrow [0 \quad 1] \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{r\theta} & \sigma_{\theta\theta} \end{bmatrix} = [\sigma_{r\theta} \quad \sigma_{\theta\theta}] = [0 \quad 0] \checkmark$$

(b) To check that force P acts @ origin consider semicircular region & show it is in equilibrium

Internal traction on S

equilibrium for this region

$$\Sigma \underline{F} = \underline{0} \Rightarrow P\underline{e}_1 + \int_S \underline{t}\, dA = 0$$



S

$$\underline{t} = \underline{n}\,\sigma = [1 \quad 0] \begin{bmatrix} \sigma_{rr} & \sigma_{\theta r} \\ \sigma_{\theta r} & \sigma_{\theta\theta} \end{bmatrix}$$

$$\Rightarrow \underline{t} = [\sigma_{rr} \quad \sigma_{\theta\theta}] = -\frac{2P}{\pi r}\cos\theta\, \underline{e_r}$$

Note $\underline{e_r} = \cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2}$

$$\Rightarrow \sum \underline{F} = 0 \Rightarrow P\underline{e_1} + \int_{-\pi/2}^{\pi/2} -\frac{2P}{\pi r}\cos\theta \left(\cos\theta\, \underline{e_1} + \sin\theta\, \underline{e_2}\right) r\, d\theta$$

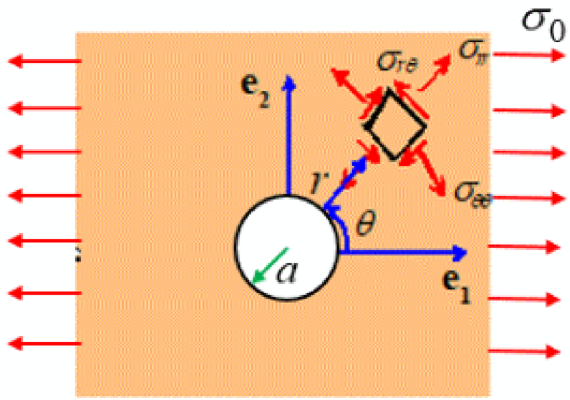
$$\Rightarrow P\underline{e_1} - P\underline{e_1} = \underline{0} \Rightarrow \text{satisfy equilibrium}$$

Note we can integrate point force solution to find solutions for any pressure distribution

Important Elasticity Problems

Cylindrical hole in a large elastic solid

Airy Function $\phi = \frac{\sigma_0}{4}\left(-2a^2\log(r)+r^2\right)+\frac{\sigma_0}{4}\left(+2a^2-r^2-\frac{a^4}{r^2}\right)\cos 2\theta$



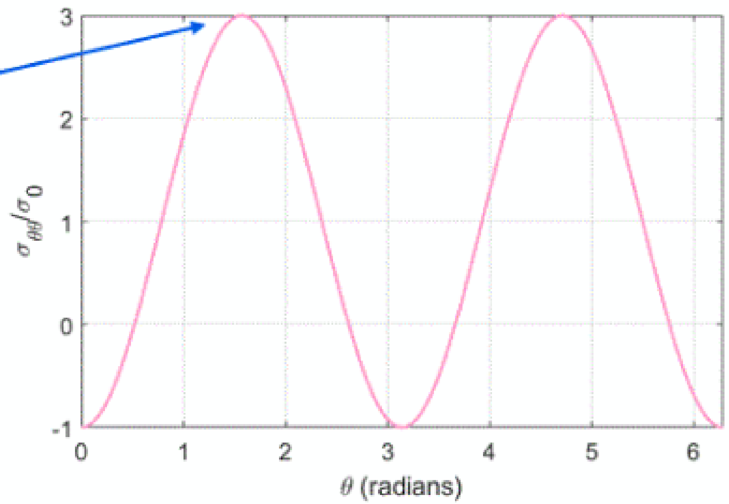
$$\sigma_{rr} = \frac{\sigma_0}{2}\left(1-\frac{a^2}{r^2}\right)\left(1+\cos 2\theta - 3\frac{a^2}{r^2}\cos 2\theta\right)$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1+\frac{a^2}{r^2}\right)-\frac{\sigma_0}{2}\cos 2\theta\left(1+3\frac{a^2}{r^2}\right)$$

$$\sigma_{r\theta} = -\frac{\sigma_0}{2}\sin 2\theta\left(1+2\frac{a^2}{r^2}-3\frac{a^4}{r^4}\right)$$

Circumferential stress around hole

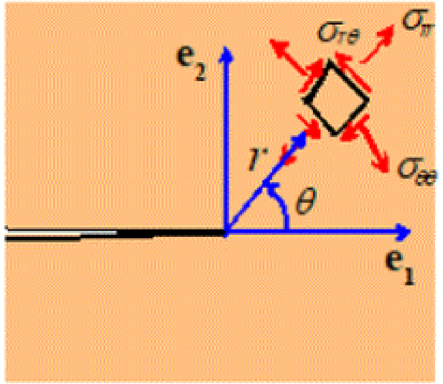
Stress concentration factor = 3



Important Elasticity Problems

Stresses near the tip of a crack:

Airy Function $$\phi=\frac{K_I}{3\sqrt{2\pi}}r^{3/2}\left(\cos 3\theta/2+3\cos\theta/2\right)-\frac{K_{II}}{\sqrt{2\pi}}r^{3/2}\left(\sin 3\theta/2+\sin\theta/2\right)$$



K_I, K_{II} are ‘stress intensity factors’ that depend on geometry of solid and how it is loaded

$$\sigma_{rr}=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{3\theta}{2}\right)+\frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right)$$

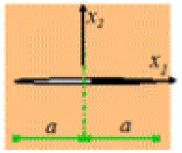
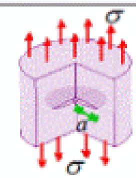
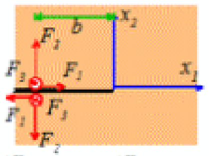
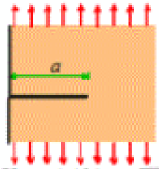
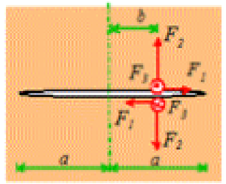
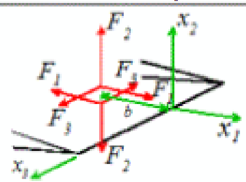
$$\sigma_{\theta\theta}=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{3\theta}{2}\right)-\frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{3}{4}\sin\frac{\theta}{2}+\frac{3}{4}\sin\frac{3\theta}{2}\right)$$

$$\sigma_{r\theta}=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{3\theta}{2}\right)+\frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2}+\frac{3}{4}\cos\frac{3\theta}{2}\right)$$

$$\sigma_{11}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)-\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)$$

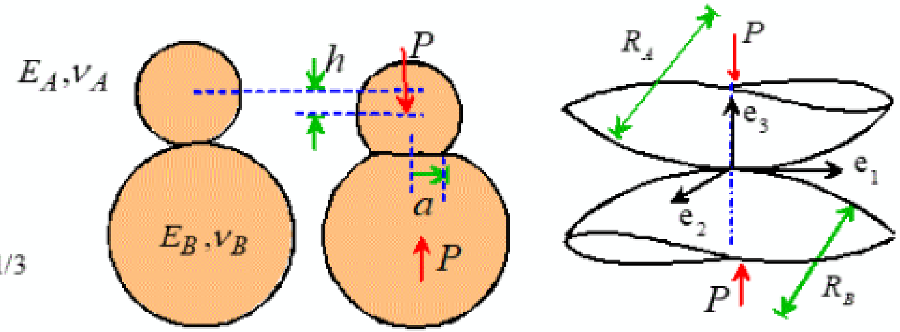
$$\sigma_{22}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_{12}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}+\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

A short table of stress intensity factors	
 $K_I=\sigma_{22}^{\infty}\sqrt{\pi a}$ $K_{II}=\sigma_{12}^{\infty}\sqrt{\pi a}$ $K_{III}=\sigma_{32}^{\infty}\sqrt{\pi a}$	 $K_I=2\sigma^{\infty}\sqrt{a/\pi}$
 $K_I=\frac{2F_2}{\sqrt{2\pi b}}$ $K_{II}=\frac{2F_1}{\sqrt{2\pi b}}$ $K_{III}=\frac{2F_3}{\sqrt{2\pi b}}$	 $K_I=1.1215\sigma\sqrt{\pi a}$
 $K_I=\frac{F_2}{\sqrt{\pi a}}f\left(\frac{b}{a}\right)$ $K_{II}=\frac{F_1}{\sqrt{\pi a}}f\left(\frac{b}{a}\right)$ $K_{III}=\frac{F_3}{\sqrt{\pi a}}f\left(\frac{b}{a}\right)$ $f(\xi)=\sqrt{(1+\xi)/(1-\xi)}$	 $K_I=\frac{4F_2}{(2\pi b)^{3/2}}f\left(\frac{x_3}{b}\right)$ $K_{II}=\frac{4F_1}{(2\pi b)^{3/2}}f\left(\frac{x_3}{b}\right)$ $K_{III}=\frac{4F_3}{(2\pi b)^{3/2}}f\left(\frac{x_3}{b}\right)$ $f(\xi)=1/\left(1+\xi^2\right)$

Important Elasticity Problems

Contact between two spheres:



Contact Radius $a = \left(\frac{3PR^*}{4E^*} \right)^{1/3}$

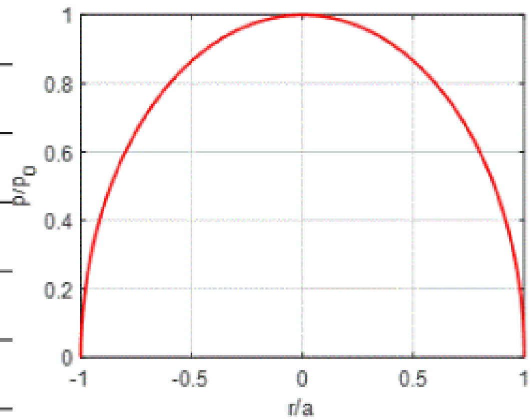
Displacement of centers $h = \frac{a^2}{R^*} = \left(\frac{9P^2}{16R^* E^{*2}} \right)^{1/3}$

$$E^* = \frac{E_A E_B}{(1-\nu_A^2)E_B + (1-\nu_B^2)E_A} \quad R^* = \frac{R_A R_B}{R_A + R_B}$$

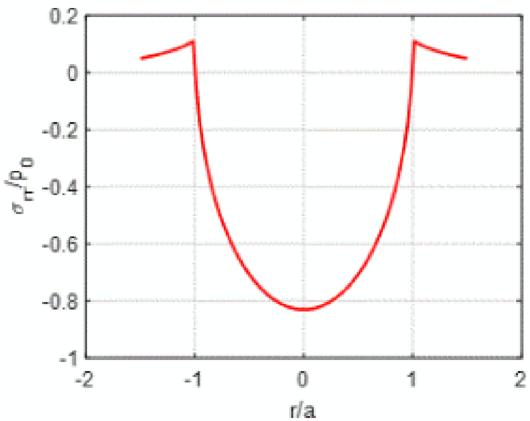
Contact Pressure $p(r) = p_0 \sqrt{a^2 - r^2}$

$$p_0 = \left(\frac{3P}{2\pi a^2} \right) = \left(\frac{6PE^{*2}}{\pi^3 R^{*2}} \right)^{1/3}$$

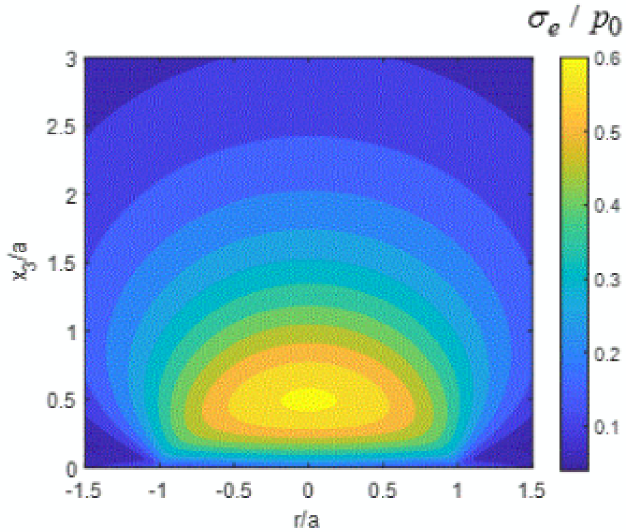
Contact Pressure



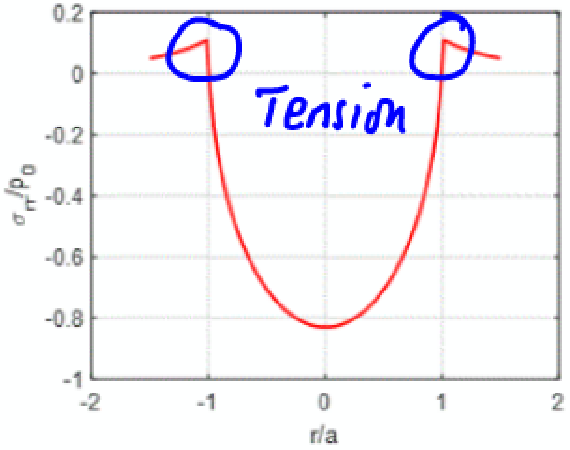
Radial stress on surface

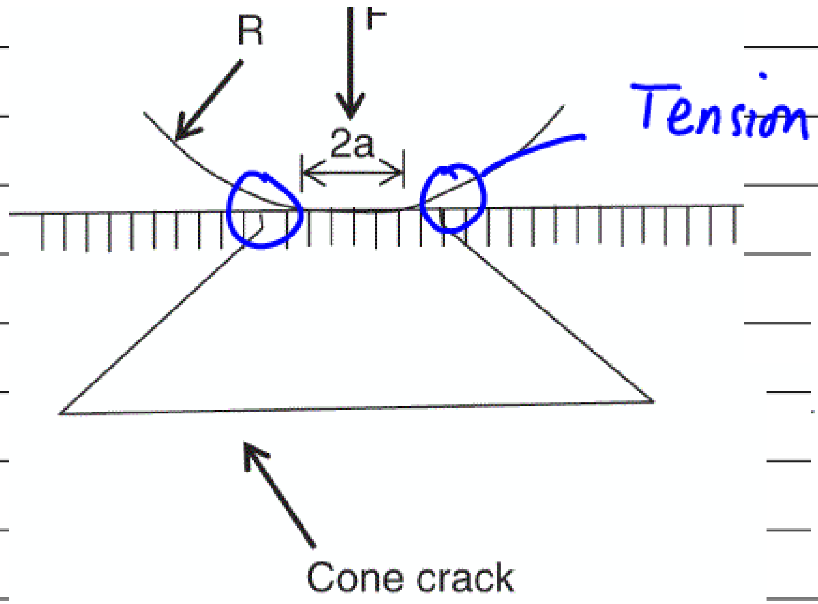


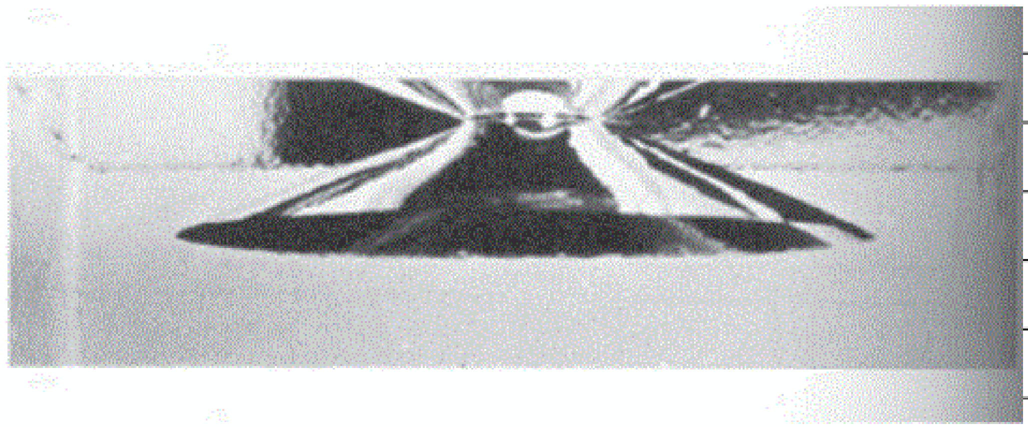
Von-Mises stress (upper sphere)



Tensile stresses on surface cause "cone cracks" in brittle materials

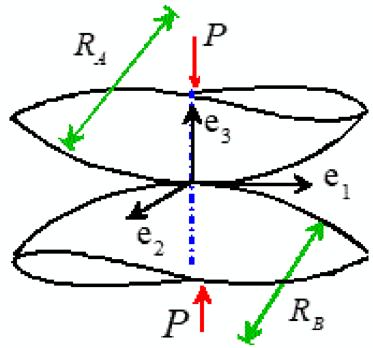






Important Elasticity Problems

Contact between two spheres:



Stresses under contact (for upper sphere)

$$\sigma_{11} = \frac{p_0}{a}\left[\phi + \frac{1}{r^2}\left\{\frac{x_2^2 - x_1^2}{r^2}\left((1-\nu)Nx_3 - \frac{1-2\nu}{3}\left(NS + 2AN + a^3\right) - \nu M x_3 a\right) - N(x_1^2 + 2\nu x_2^2) - \frac{M x_1^2 x_3 a}{S}\right\}\right]$$

$$\sigma_{22} = \frac{p_0}{a}\left[\phi + \frac{1}{r^2}\left\{\frac{x_1^2 - x_2^2}{r^2}\left((1-\nu)Nx_3 - \frac{1-2\nu}{3}\left(NS + 2AN + a^3\right) - \nu M x_3 a\right) - N(x_2^2 + 2\nu x_1^2) - \frac{M x_2^2 x_3 a}{S}\right\}\right]$$

$$\sigma_{33} = \frac{p_0}{a}\left(-N + \frac{a x_3 M}{S}\right) \qquad \sigma_{13} = -\frac{x_3 x_1 p_0}{a}\left(\frac{N}{S} - \frac{x_3 H}{G^2 + H^2}\right) \qquad \sigma_{23} = -\frac{x_3 x_2 p_0}{a}\left(\frac{N}{S} - \frac{x_3 H}{G^2 + H^2}\right)$$

$$\sigma_{12} = \frac{p_0 x_1 x_2}{a r^4}\left[(1-2\nu)\left\{-N r^2 + \frac{2}{3}N(S + 2A) - x_3(x_3 N + aM) + \frac{2}{3}a^3\right\} + x_3\left\{-\frac{aMr^2}{S} - x_3 N + aM\right\}\right]$$

where

$$r = \sqrt{x_1^2 + x_2^2} \quad A = r^2 + x_3^2 - a^2 \quad S = \sqrt{A^2 + 4a^2 x_3^2} \quad M = \sqrt{(S + A)/2} \quad N = \sqrt{(S - A)/2}$$

$$G = M^2 - N^2 + x_3 M - aN \quad H = 2MN + aM + x_3 N \quad \phi = (1+\nu)x_3 \tan^{-1}(a/M)$$

On r=0

$$\sigma_{11} = \sigma_{22} = \frac{p_0}{a}\left[(1+\nu)\left(x_3 \tan^{-1}(a/x_3) - a\right) + \frac{a^3}{2\left(a^2 + x_3^2\right)}\right] \qquad \sigma_{33} = -\frac{p_0 a^2}{a^2 + x_3^2}$$