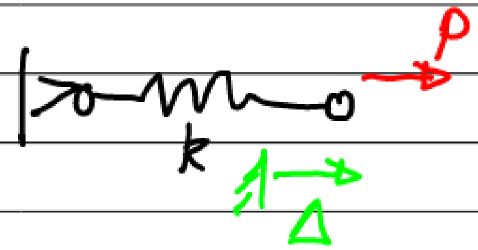
9 Analyzing elastic solids using energy

Background: we can calculate deflection of a spring by minimizing its PE



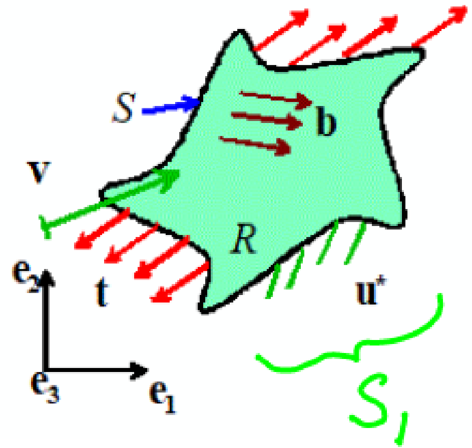
PE: $\Pi = \frac{1}{2} k \Delta^2 - P\Delta$

Minimize wrt Δ: $\frac{\partial \Pi}{\partial \Delta} = k\Delta - P = 0 \Rightarrow \Delta = \frac{P}{k}$

Same idea works for continuum

- Applications: use energy to get approximate solutions
  : energy can be used to derive FEA equations

9.1 Definition of PE for elastic solids



Assumptions : (1) Elastic solid ; small defomation
(2) No temp changes
(3) Static equilibrium
(4) Boundary conditions $\underline{u} = \underline{u}^* \text{ on } S_1$

$\underline{n}\,\sigma = \underline{t} \text{ on } S_2$

Exact solution $[\underline{u}, \varepsilon, \sigma]$

Approximation for displacement

Let $\hat{\underline{v}}(\underline{x})$ be a guess for displacement satisfying

$\hat{\underline{v}} = \underline{u}^* \text{ on } S_1$ "kinematically admissible displacement field"

Strain $\hat{\varepsilon} = [\nabla \hat{v} + (\nabla \hat{v})^T]/2$

Definition of PE

Strain energy density $\bar{U} = \frac{1}{2}\,\hat{\underline{\sigma}} \cdot \hat{\underline{\varepsilon}} = \frac{1}{2}\,\hat{\underline{\varepsilon}} \cdot \big(\underbrace{[C]}_{\text{elastic constants}}\,\hat{\underline{\varepsilon}}\big)$

$\hat{\underline{\varepsilon}} = [\,\hat{\varepsilon}_{11},\ \hat{\varepsilon}_{22},\ \hat{\varepsilon}_{33},\ 2\hat{\varepsilon}_{12},\ 2\hat{\varepsilon}_{13},\ 2\hat{\varepsilon}_{23}\,]$

$\hat{\sigma} = [\,\sigma_{11},\ \sigma_{22},\ \sigma_{33},\ \sigma_{12}\ \text{etc}\,]$

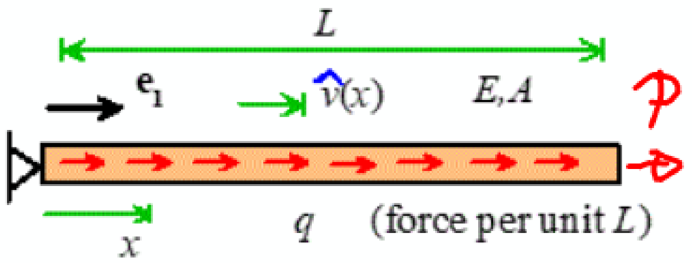
Define PE

$$\Pi(\hat{\underline{v}}) = \int_V \bar{U}\, d\bar{V} - \int_V \rho\, \underline{b} \cdot \hat{\underline{v}}\, d\bar{V} - \int_{S_2} \underline{t} \cdot \hat{\underline{v}}\, dA$$

Example: 1D bar

$\hat{\varepsilon} = \dfrac{d\hat{v}}{dx}$ $\quad \sigma = E\hat{\varepsilon}$ $\quad \bar{U} = \frac{1}{2}\sigma\hat{\varepsilon}$

$= \frac{1}{2}E\left(\dfrac{d\hat{v}}{dx}\right)^2$



$\sigma_{11} \neq 0$ all others 0

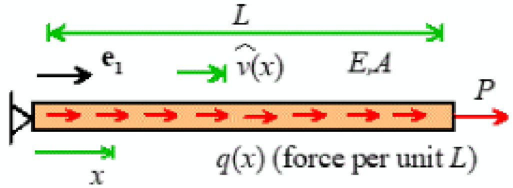
Hence $\Pi = \int_0^L \frac{1}{2} A E \left(\frac{d\hat{u}}{dx}\right)^2 dx - \int_0^L q\hat{u}\, dx - P\hat{u}(L)$

Can calculate Π for many other geometries

Useful formulas for potential energy

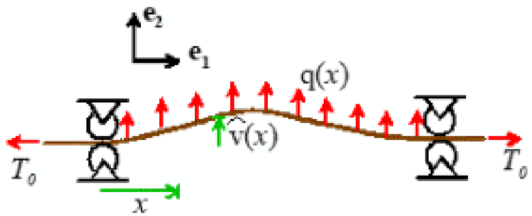
1-D axially loaded bar

$$\Pi = \int_0^L \frac{1}{2} EA \left(\frac{d\hat{v}}{dx} \right)^2 dx - \int_0^L q(x)\hat{v}(x)dx - Pv(L)$$



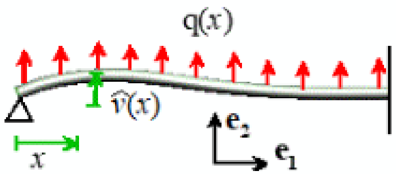
1-D Tensioned cable

$$\Pi = T_0 \int_0^L \frac{1}{2} \left(\frac{d\hat{v}}{dx} \right)^2 dx - \int_0^L q(x)\hat{v}(x)dx$$



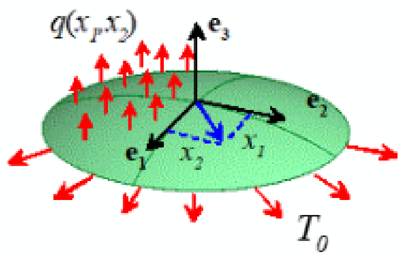
1-D Euler-Bernoulli beam

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{d^2\hat{v}}{dx^2} \right)^2 dx - \int_0^L q(x)\hat{v}(x)dx$$



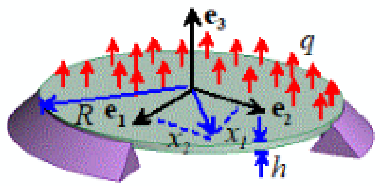
2-D biaxially stretched membrane

$$\Pi = \int_A \frac{1}{2} T_0 \left[\left(\frac{\partial \hat{v}}{\partial x_1} \right)^2 + \left(\frac{\partial \hat{v}}{\partial x_2} \right)^2 \right] dA - \int_A q(x_1,x_2)\hat{v}(x_1,x_2)dA$$



2-D Kirchhoff plate

$$\Pi = \frac{Eh^3}{12(1-\nu)} \int_A \frac{1}{2} \left[\left(\frac{\partial^2 \hat{v}}{\partial x_1^2} + \frac{\partial^2 \hat{v}}{\partial x_2^2} \right)^2 - 2(1-\nu) \left(\frac{\partial^2 \hat{v}}{\partial x_1^2} \frac{\partial^2 \hat{v}}{\partial x_2^2} - \left(\frac{\partial^2 \hat{v}}{\partial x_1 \partial x_2} \right)^2 \right) \right] dA - \int_A q(x_1,x_2)\hat{v}(x_1,x_2)dA$$

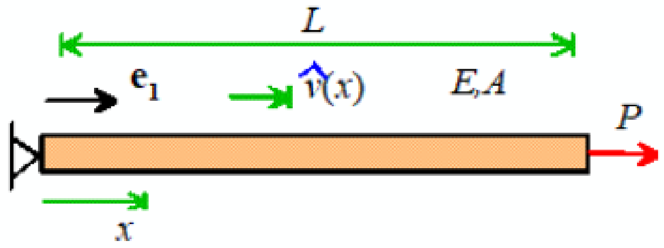


9.3 Principle of minimum potential energy

Among all possible guesses $\hat{v}$ the exact solution [satisfies stress equilibrium] has minimum PE

$$\Pi(\hat{v}) \geq \Pi(\underline{u})$$

Example: 1D bar with axial load



Exact solution $\sigma = \frac{P}{A}$ $\quad \varepsilon = \sigma / E$

$$\Rightarrow \frac{du}{dx} = \varepsilon \Rightarrow u = \frac{P}{AE} x$$

$$\Pi(u) = \int_0^L \frac{1}{2} EA \left(\frac{du}{dx}\right)^2 dx - P \frac{PL}{AE} = -\frac{P^2 L}{2AE}$$

Guess $\hat{u} = \beta x$ (β some constant)

$$\Pi(\hat{u}) = \int_0^L \frac{1}{2} EA \left(\frac{d\hat{u}}{dx}\right) dx - P\beta L = \frac{1}{2} EA\beta^2 - PL\beta$$

Hence $\Pi(\hat{u}) - \Pi(u) = \frac{1}{2} LEA\beta^2 - PL\beta + \frac{P^2 L}{2AE}$

$$= \underbrace{\frac{1}{2} EAL\left(\beta - \frac{P}{AE}\right)^2}_{\geqslant 0}$$

Hence $\Pi(\hat{u}) \geqslant \Pi(u)$

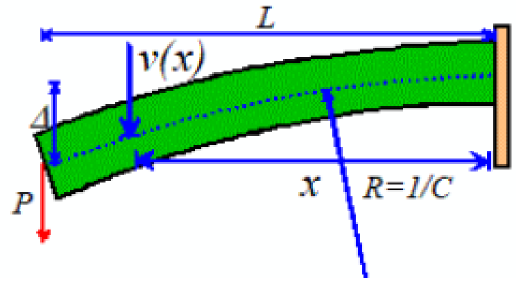
Same idea used in general proof.

9.4 Applications of energy I : Estimating stiffness

Consider an example

Example: The beam has modulus *E* and moment of inertia *I*, and has stiffness *k* such that $P = k\Delta$ $\quad (k_{exact} = 3EI/L^3)$

Using $\hat{v} = Cx^2/2$ as a guess for the displacement, estimate *k*



Procedure:

(1) Find $\Pi(u)$ in terms of k, P $\qquad \Pi = \frac{1}{2}k\Delta^2 - P\Delta$

$$\Delta = P/k \Rightarrow \Pi(u) = -P^2/2k$$

(2) Find best (lowest) possible guess for $\Pi(\hat{v})$

Using given guess $\Pi(\hat{v}) = \int_0^L \frac{1}{2} EI \left(\frac{d^2\hat{v}}{dx^2}\right)^2 dx - P\hat{v}(L)$

$$\Rightarrow \Pi(\hat{v}) = \int_0^L \frac{EI}{2} C^2 dx - P \frac{CL^2}{2} = \frac{EI}{2} L C^2 - \frac{PCL^2}{2}$$

Minimize wrt C for best guess

$$\Rightarrow \frac{\partial \Pi}{\partial C} = 0 = EILC - \frac{PL^2}{2} = 0 \Rightarrow C = \frac{PL}{2EI}$$

$$\Rightarrow \Pi(\hat{v}) = -\frac{P^2L^3}{8EI}$$

$$\Pi(\hat{v}) \geq \Pi(u) \Rightarrow -\frac{P^2L^3}{8EI} \geq -\frac{P^2}{2k}$$

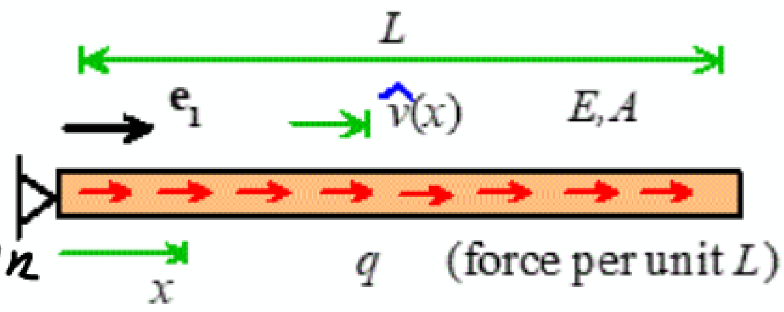
$$\Rightarrow \boxed{k \leq \frac{4EI}{L^3}}$$

"close enough for government work"

9.5 Applications 2: "Rayleigh - Ritz" approximation

General method for getting approximate solutions

Consider example (1D bar with body force)



Procedure: (1) Find a general approximation for $\hat{v}$

$$\hat{v} = \sum_{i=1}^{N} a_i f_i(x)$$

where $f_i(x)$ are "basis functions" eg $f_i = x^{i-1}$

$f_i = \sin(\pi (i-1) x / L)$

a_i : set of unknown coefficients TBD

To find a_i : (1) Eliminate some subset of a_i using boundary condition for $\hat{v}$

- get new $\hat{v}$ with fewer unknowns

(2) Find $\Pi(a_i)$; minimize

$$\frac{\partial \Pi}{\partial a_i} = 0 \Leftarrow \text{several eqs}$$

- solve for a_i

For our example try $\hat{v} = a_1 + a_2 x$

(1) $\hat{v}(x) = 0$ @ $x = 0$ $\Rightarrow a_1 = 0$

$$\Rightarrow \hat{v} = a_2 x$$

(2) $\Pi = \int_0^L \frac{EA}{2}\left(\frac{d\hat{v}}{dx}\right)^2 dx - \int_0^L q\hat{v}\, dx$

$\frac{EA}{2} a_2^2 L - \frac{1}{2} a_2 L^2 q$

Minimize $\Rightarrow \frac{\partial \Pi}{\partial a_2} = 0 \qquad EA\, a_2 L - \frac{1}{2} L^2 q = 0$

$\Rightarrow a_2 = \frac{qL}{2EA}$

$$\boxed{\hat{v} = \frac{qL}{2EA} x}$$

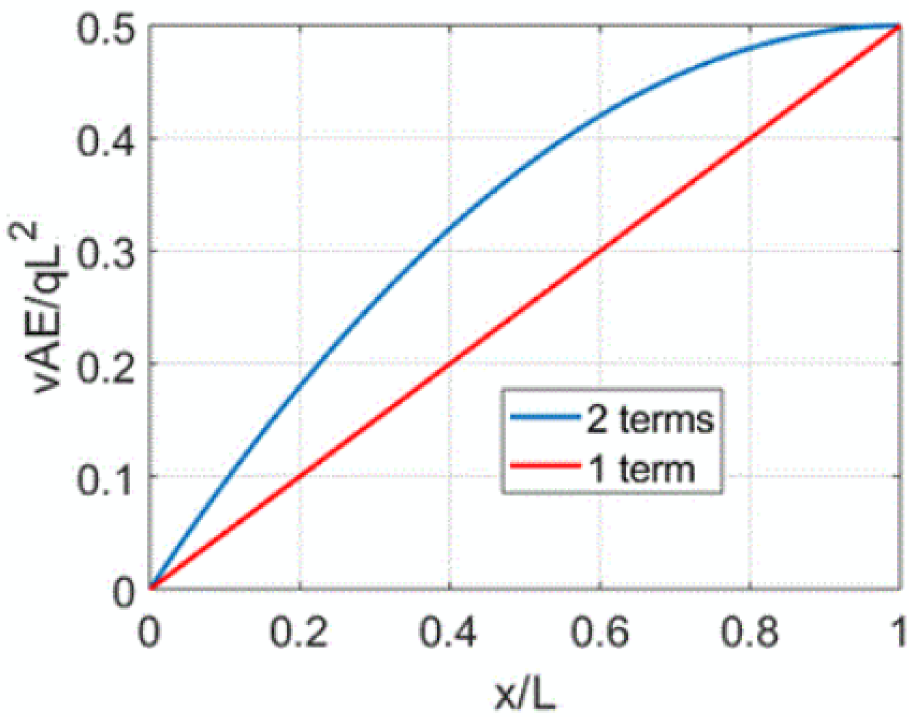
Try better guess $a_1 + a_2 x + a_3 x^2$

- have MATLAB do algebra

```
syms v x a0 a1 a2 L E A q PI
v = a0 + a1*x + a2*x^2;    % Approximation for displacement
eq1 = subs(v,x,0)==0;  % Boundary condition v=0 at x=0
a0sol = solve(eq1,a0); % Solve for a0 (in terms of the other as)
v = subs(v,a0,a0sol);  % Eliminate a0
PI = int(E*A/2*diff(v,x)^2 - q*v ,x,[0,L]); % Potential energy
eq2 = diff(PI,a1)==0;  % d PE/da1 = 0
eq3 = diff(PI,a2)==0;  % d PE/da2 = 0
[as1,as2] = solve([eq2,eq3],[a1,a2]); % Solve the equations
v = subs(v,[a1,a2],[as1,as2])  % Substitute solution into v
```

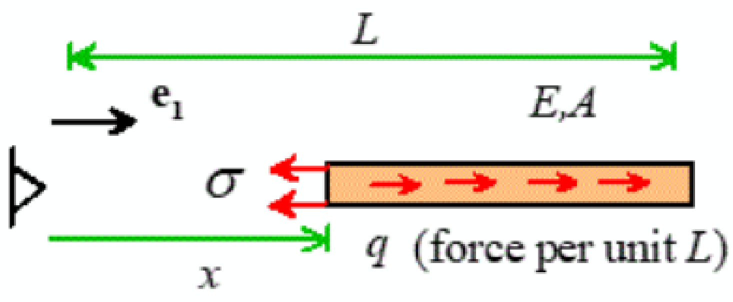
v =

$$\frac{L\,q\,x}{A\,E} - \frac{q\,x^2}{2\,A\,E}$$



What is the correct solution

Solve governing eq's exactly



(1) Equilibrium (method of sections)

$$\sigma A = q(L-x)$$

$$\Rightarrow \sigma = q(L-x)/A$$

(2) Stress-strain law $\varepsilon = \sigma / E = q(L-x)/AE$

(3) Strain-displacement $\varepsilon = \frac{du}{dx} \Rightarrow u = \int_0^x \varepsilon\, dx$

$$\Rightarrow u = \frac{qLx}{AE} - \frac{qx^2}{2AE}$$

2 terms gives exact sol

In general if approximation can describe exact solution with finite # terms it will give the exact sol