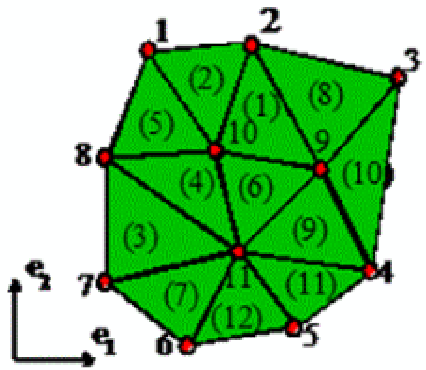
Review – Simple FEA for plane linear elasticity

- **Approach**: compute displacement field in an elastic solid by
  - Interpolating displacement field
  - Calculating total potential energy of solids in terms of discrete displacements
  - Minimize potential energy



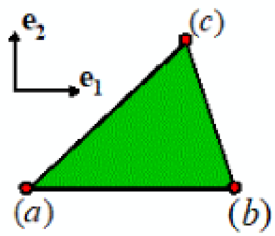
- **Interpolation** – constant strain triangles

$$u_i(x_1,x_2) = u_i^{(a)} N^a(x_1,x_2) + u_i^{(b)} N^b(x_1,x_2) + u_i^{(c)} N^c(x_1,x_2)$$

$$N^a(x_1,x_2) = \frac{\left(x_2 - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1 - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}{\left(x_2^{(a)} - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1^{(a)} - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}$$

$$N^b(x_1,x_2) = \frac{\left(x_2 - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1 - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}{\left(x_2^{(b)} - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1^{(b)} - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}$$

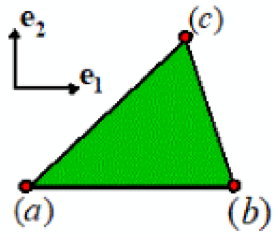
$$N^c(x_1,x_2) = \frac{\left(x_2 - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1 - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}{\left(x_2^{(c)} - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1^{(c)} - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}$$



- **Potential Energy** $\Pi = \int_A U dA - \int_{S_2} \mathbf{t}^* \cdot \mathbf{u} ds$

Review – Strain energy density in an element

$$\underline{\varepsilon} = [B]\underline{u}^{\text{element}} \equiv \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 & \frac{\partial N_b}{\partial x_1} & 0 & \frac{\partial N_c}{\partial x_1} & 0 \\ 0 & \frac{\partial N_a}{\partial x_2} & 0 & \frac{\partial N_b}{\partial x_2} & 0 & \frac{\partial N_c}{\partial x_2} \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} & \frac{\partial N_b}{\partial x_2} & \frac{\partial N_b}{\partial x_1} & \frac{\partial N_c}{\partial x_2} & \frac{\partial N_c}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1^{(a)} \\ u_2^{(a)} \\ u_1^{(b)} \\ u_2^{(b)} \\ u_1^{(c)} \\ u_2^{(c)} \end{bmatrix}$$



$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

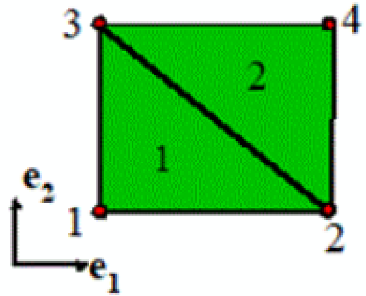
$$W^{\text{element}} = \frac{1}{2}\underline{u}^{\text{element}T}\left(A_{element}[B]^T[D][B]\right)\underline{u}^{\text{element}}$$

$$W^{\text{element}} = \frac{1}{2}\underline{u}^{\text{element}T}[K^{\text{element}}]\underline{u}^{\text{element}}$$

$$[K^{\text{element}}] = A_{element}[B]^T[D][B]$$ Symmetric 6x6 matrix

10.3 Compute total strain energy

Illustrate with 2 element mesh



$$W = \sum_{elements} W^{elem} = \sum_{element} \frac{1}{2} \underline{u}^{elem} \cdot \left\{ [k^{elem}]\, \underline{u}^{elem} \right\}$$

We can't evaluate sum easily in this form

Re-write in terms of "global" displacement vector

$$\underline{u}^g = [u_1^1 \;\; u_2^1 \;\; u_1^2 \;\; u_2^2 \;\; \ldots \;\; u_1^4 \;\; u_2^4]$$

(for N nodes 2N long vector)

We can expand each $[k^{elem}]$ into $2N \times 2N$ matrices by adding zeros into rows/cols corresponding to DOF not part of element eg for element # 1

$$\underset{\sim}{u}^g \cdot \left\{ \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & \cdots & k_{16}^{(1)} & 0 & 0 \\ \vdots & & & & & \\ k_{61}^{(1)} & & & & & \\ 0 & 0 & & \cdots & & 0 \\ 0 & 0 & & & & \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ \vdots \end{bmatrix} \right\}$$

We can now sum all $2N \times 2N$ matrices

$$W = \frac{1}{2} \underset{\sim}{u}^g \cdot \left\{ \underbrace{\sum_{elem} [k^{el}]}_{\text{"Global" stiffness } [K]} \underset{\sim}{u}^g \right\}$$

"Global" stiffness $[K]$

$2N \times 2N$ symmetric matrix

Written out in full

$$W=\frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} \end{bmatrix}\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & \cdots & k_{16}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & & \\ \vdots & & \ddots & \\ k_{61}^{(1)} & & & k_{66}^{(1)} \end{bmatrix}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}+\frac{1}{2}\begin{bmatrix} u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & u_1^{(4)} & u_2^{(4)} \end{bmatrix}\begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & \cdots & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & & \\ \vdots & & \ddots & \\ k_{61}^{(2)} & & & k_{66}^{(2)} \end{bmatrix}\begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

Expanded

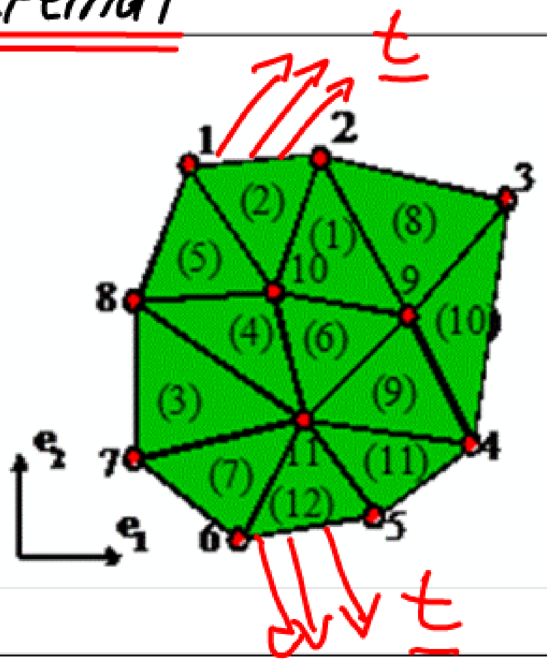
$$W=\frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & u_1^{(4)} & u_2^{(4)} \end{bmatrix}\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & & \cdots & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & & 0 & 0 \\ & & k_{33}^{(1)} & k_{34}^{(1)} & & & 0 & 0 \\ & & k_{43}^{(1)} & k_{44}^{(1)} & & & 0 & 0 \\ & & k_{53}^{(1)} & & & & 0 & 0 \\ & & \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}+\frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & u_1^{(4)} & u_2^{(4)} \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & k_{11}^{(2)} & k_{12}^{(2)} & & & & \\ 0 & 0 & k_{21}^{(2)} & k_{22}^{(2)} & & & & \\ 0 & 0 & k_{31}^{(2)} & & & & & \\ 0 & 0 & & & & \ddots & & \\ 0 & 0 & \vdots & & & & & k_{56}^{(2)} \\ 0 & 0 & & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

Final form

$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & u_1^{(4)} & u_2^{(4)} \end{bmatrix} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & \cdots & & \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & & \\ & & k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & & & \\ & & k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & & & \\ & & k_{53}^{(1)} + k_{31}^{(2)} & & & & \\ & & & & \ddots & & \\ & & \vdots & & & & k_{56}^{(2)} \\ & & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

In actual code we don't store expanded $[k^{elem}]$ but just add each element stiffness to correct row / column of global stiffness

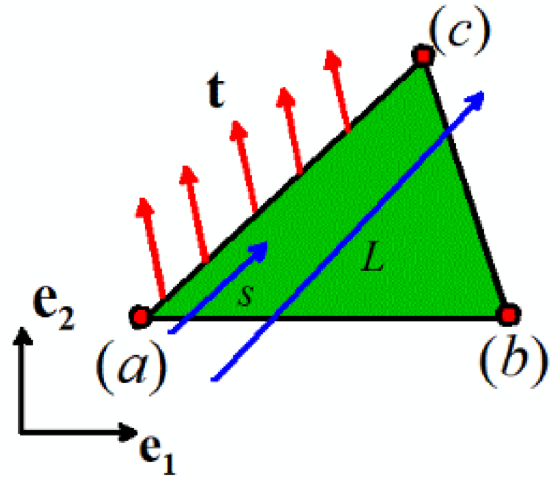
10.7 Calculating potential energy of external tractions



Assume any one element has a constant traction (can be different on different element)

Focus on one element face

Recall $\hat{v}$ varies linearly with position in element



Hence $\underline{\hat{v}} = \underline{U}^{(a)}\left(1 - \frac{s}{L}\right) + \underline{U}^{(c)} s/L$

(Linear variation from (a) → (c))

Now consider

$$-\int_{S_2} \underline{t} \cdot \hat{\underline{\nu}} \, dA = -\int_0^L [t_1 \; t_2] \cdot \left\{ [u_1^a \; u_2^a] (1 - s/L) + [u_1^c \; u_2^c] \, s/L \right\} ds$$

$$= -\underbrace{\frac{L}{2} [t_1 \; t_2 \; t_1 \; t_2]}_{\underline{r}^{face}} \cdot \underbrace{[u_1^a \; u_2^a \; u_1^c \; u_2^c]}_{\underline{u}^{face}}$$

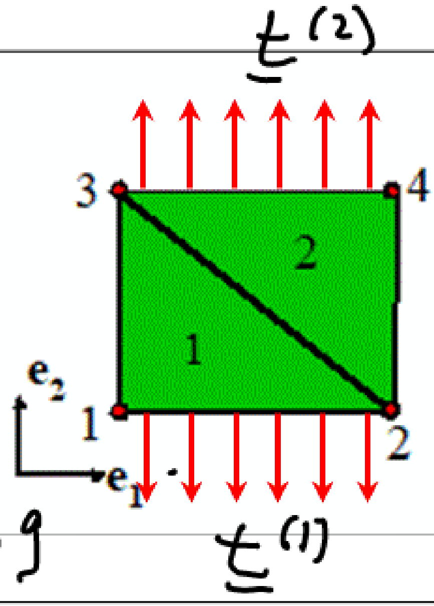
$$\Rightarrow -\int \underline{t} \cdot \hat{\underline{\nu}} \, dA = -\underline{r}^{face} \cdot \underline{u}^{face}$$

Need to sum contribution from all el faces

$$-\int \hat{\underline{t}} \cdot \hat{\underline{\nu}} \, dA = -\sum_{faces} \underline{r}^{face} \, \underline{u}^{face}$$

Need to re-write in terms of $\underline{u}^g$

eg for 2 element mesh



$$\underbrace{[r_1^{(1)} \quad r_2^{(1)} \quad r_1^{(2)} \quad r_2^{(2)} \quad 0 \quad 0 \quad 0 \quad 0]}_{\text{from face (1)}} \cdot \underline{u}^g$$

$$+ [0 \quad 0 \quad 0 \quad 0 \quad r_1^{(3)} \quad r_2^{(3)} \quad r_1^{(4)} \quad r_2^{(4)}] \cdot \underline{u}^g$$

We can now sum the expanded force vectors

$$\rightarrow \underbrace{\sum \underline{r}^{face}}_{\text{"global" force vector } \underline{r}^g} \cdot \underline{u}^g$$

(2N long vector)

(unloaded faces contribute zero to $\underline{r}^g$)

10.8 Minimizing Π

We have $\Pi = \frac{1}{2} \underline{u}^g \cdot \{ [K] \underline{u}^g \} - \underline{r}^g \cdot \underline{u}^g$

Can minimize using index notation

$$\Pi = \frac{1}{2} u_i^g K_{ij} u_j^g - r_i^g u_i^g$$

To minimize set $\frac{\partial \Pi}{\partial u_k^g} = 0$ for $k = 1 \dots 2N$

$$\frac{\partial \Pi}{\partial u_k} = \frac{1}{2} \{ \underbrace{\delta_{ik} K_{ij}}_{K_{kj}} u_j^g + u_i^g \underbrace{K_{ij} \delta_{jk}} \} - \underbrace{r_i^g \delta_{ik}}_{r_k^g}$$

$u_i^g K_{ik} = u_i^g K_{ki}$

$[K]$ symmetric

$= K_{kj} u_j^g$

$$= K_{kj} u_j^g - r_k^g = [K] \underline{u}^g - \underline{r}^g = 0$$

System of linear equations for $\underline{u}^g$

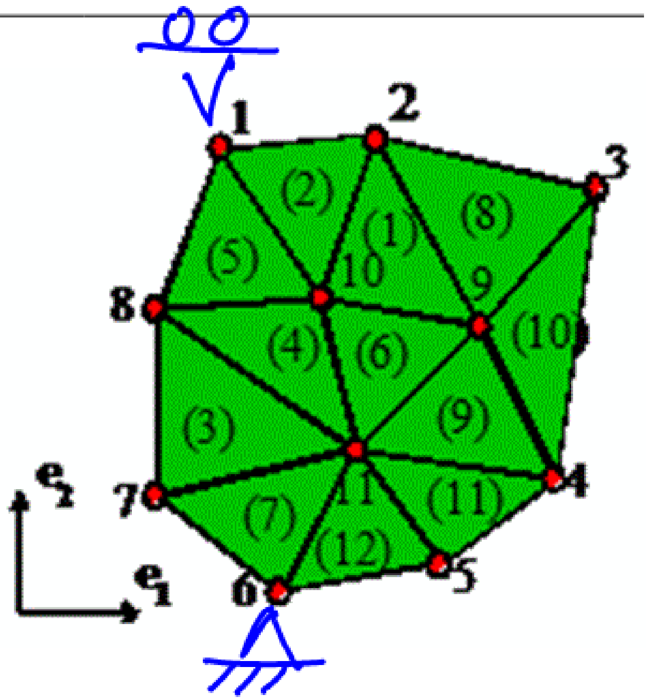
We still need to prescribe displacements at some nodes

10.9 Enforcing prescribed displacements

We can modify our equation system

$$[K]\,\underline{u}^g = \underline{r}^g$$

eg to enforce $u_2^{(1)} = \Delta$ (node #1 has $u_2 = \Delta$)



Node # 1 corresponds to 1st two rows in equation system

Replace 2nd equation with $u_2^2 = \Delta$

eg if original system is

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2N} \\ \vdots & & & \\ k_{2N\,1} & & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r_1^1 \\ r_2^1 \\ r_1^2 \\ r_2^2 \\ \vdots \end{bmatrix} \Leftarrow \text{Replace this row with } u_2^2 = \Delta$$

becomes

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ 0 & 1 & \cdots & 0 \\ k_{21} & & & \\ \vdots & & & k_{2N\,2N} \end{bmatrix} = \begin{bmatrix} r_1^1 \\ \Delta \\ r_1^2 \\ r_2^2 \\ \vdots \end{bmatrix}$$

Repeat for all nodes with known displacements

10.10 Implementing FEA code

(1) Read data defining problem (GUI or text file)

(a) Material props E, ν

(b) Nodal coords

(c) Element connectivity

(d) List of nodes with prescribed DOF

(e) List of el faces with nonzero tractions

(2) Loop over elements; find $[K^{el}]$ and add to $[K]$

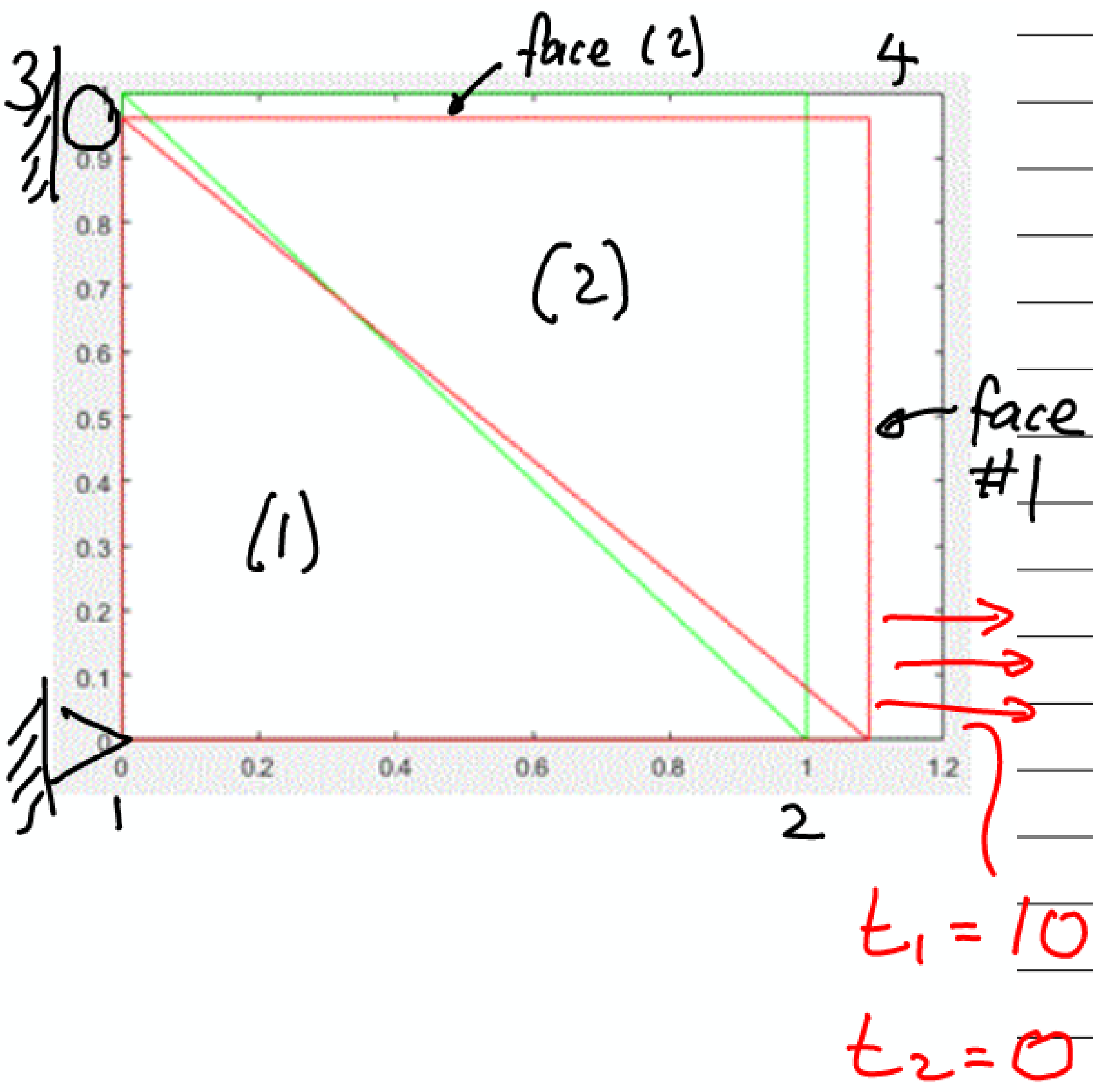
(3) Loop over loaded faces $\underline{r}^{face}$; add to $\underline{r}^{g}$

(4) Modify $[K]$ $\underline{r}^{g}$ to prescribe displacements

(5) Solve $[K]\underline{u}^{g} = \underline{r}^{g}$ for $\underline{u}^{g}$

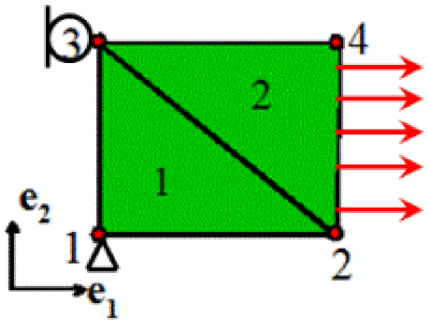
(6) Post-processing to find strain, stress etc

```
Material_Props:
    Young's_modulus:   100.
    Poissons_ratio:  0.3
No._nodes:             4
Nodal_coords:
    0.0   0.0
    1.0   0.0
    0.0   1.0
    1.0   1.0
No._elements:                    2
Element_connectivity:
    1 2 3
    2 4 3
No._nodes_with_prescribed_DOFs:  3
Node_#, DOF#, Value:
   1 1 0.0
   1 2 0.0
   3 1 0.0
No._elements_with_prescribed_loads: 1
Element_#, Face_#, Traction_components
  2 1 10.0 0.0
```

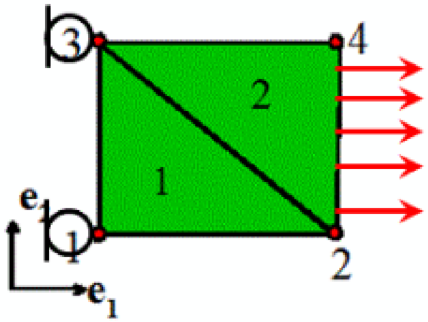


Improper constraints lead to singular stiffness matrix

```
eigenvecs =

         0         0   -0.3152   -0.4728         0   -0.3152    0.4781    0.5908
         0         0    0.6043   -0.4559         0    0.6043    0.0433    0.2449
         0         0    0.0866    0.6137         0    0.0866   -0.2079    0.7518
         0         0   -0.7071   -0.0000         0    0.7071   -0.0000    0.0000
         0         0   -0.1673   -0.4381         0   -0.1673   -0.8522    0.1606
    0.4779         0    0.4907    0.5123         0    0.0000   -0.0080    0.5179
         0    0.4779    0.0000    0.5123         0    0.4907   -0.0080    0.5179
         0         0   -0.0039   -0.5103    0.4759   -0.0039    0.4928   -0.5198


eigenvals =

  199.8263         0         0         0         0         0         0         0
         0  125.4412         0         0         0         0         0         0
         0         0   13.8635         0         0         0         0         0
         0         0         0   38.4615         0         0         0         0
         0         0         0         0   55.0998         0         0         0
         0         0         0         0         0    1.0000         0         0
         0         0         0         0         0         0    1.0000         0
         0         0         0         0         0         0         0    1.0000
```



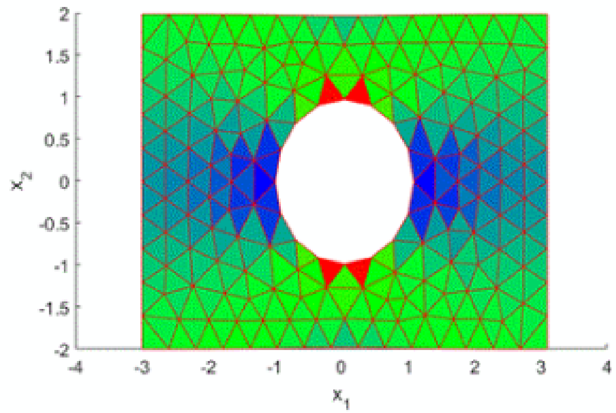

```
eigenvecs =

         0    0.3791   -0.3791   -0.3791         0   -0.4610    0.3791    0.4610
         0   -0.4523    0.3015   -0.4523         0    0.4523    0.3015    0.4523
         0   -0.5000   -0.5000    0.5000         0   -0.0000    0.5000    0.0000
         0   -0.2132   -0.6396   -0.2132         0    0.2132   -0.6396    0.2132
         0   -0.3260    0.3260    0.3260         0   -0.5361   -0.3260    0.5361
         0    0.5000    0.0000    0.5000         0    0.5000    0.0000    0.5000
    0.5638   -0.2906    0.5789    0.2929         0   -0.2984   -0.0046    0.2961
         0    0.2929   -0.0046   -0.2906    0.5638    0.2961    0.5789   -0.2984


eigenvals =

  216.4663         0         0         0         0         0         0         0
         0  153.8462         0         0         0         0         0         0
         0         0   76.9231         0         0         0         0         0
         0         0         0   48.0769         0         0         0         0
         0         0         0         0   23.9183         0         0         0
         0         0         0         0         0   -0.0000         0         0
         0         0         0         0         0         0    1.0000         0
         0         0         0         0         0         0         0    1.0000

> In fem_conststrain_triangles (line 93)
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND =
2.771117e-17.
```



Volumetric locking in near-incompressible materials

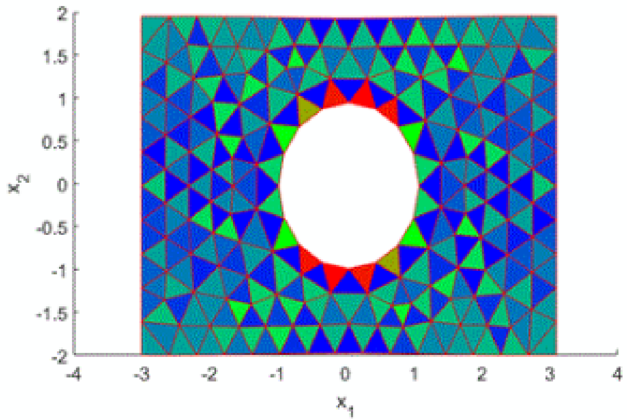
Example problem: plane strain strip with central hole

Contours show σ_{11}



Results for $\nu = 0.3$

Results for $\nu = 0.499$



Spurious pressure fluctuations – this happens because the elements become very stiff

Constant strain triangles always give incorrect results for near incompressible materials – there is no fix.

For other element types, reduced integration is used to correct volumetric locking.
Hybrid elements are specially designed to be used for near incompressible materials