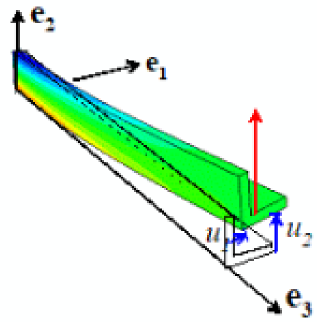
Review: Beams



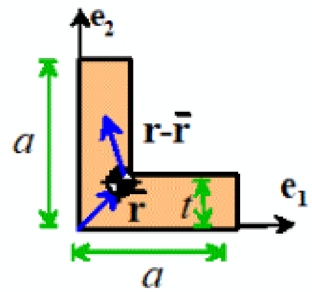
Goal: Calculate (1) Displacement of centroid $\mathbf{u}(x_3) = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$

(2) Rotation of x-section $\boldsymbol{\theta}(x_3) = \theta_1\mathbf{e}_1 + \theta_2\mathbf{e}_2 + \theta_3\mathbf{e}_3$

(3) Curvature vector $\boldsymbol{\kappa}(x_3) = d\boldsymbol{\theta} / dx_3$

Neglect twist θ_3 here, but ABAQUS will include twist

Section properties:



$$A = \int_A dA \qquad \bar{\mathbf{r}} = \frac{1}{A}\int_A (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)dA$$

$$\mathbf{I} = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix} \qquad I_{11} = \int_A (x_2 - \bar{r}_2)^2 dA \qquad I_{22} = \int_A (x_1 - \bar{r}_1)^2 dA \qquad I_{12} = \int_A (x_1 - \bar{r}_1)(x_2 - \bar{r}_2)dA$$

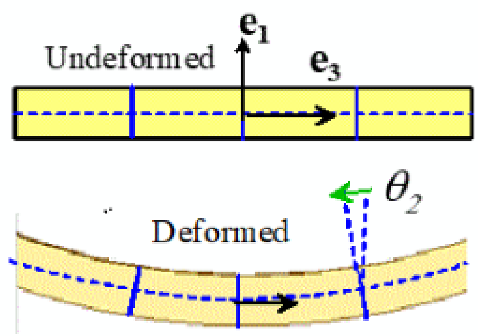
Deformation:

Euler-Bernoulli theory (no shear): $\theta_1 = -du_2 / dx_3 \qquad \theta_2 = du_1 / dx_3$

(Timoshenko theory allows x-sect to rotate relative to neutral axis)

Axial strain: $\varepsilon_{33} = \kappa_1(x_2 - \bar{r}_2) - \kappa_2(x_1 - \bar{r}_1)$

(Timoshenko beam has shear strains)



Stresses: $\sigma_{33} = E\varepsilon_{33}$

Other stresses zero in E-B beams
(Timoshenko beams have shear stresses)

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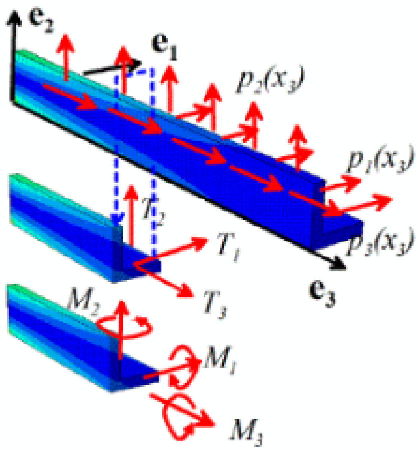
Internal Forces (forces/moments on section normal to $\mathbf{e}_3$:



Force vector $\quad \mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3$

Moment vector $\quad \mathbf{M} = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3$

$$M_1 = \int_A \sigma_{33}(x_2 - \overline{x}_2)dA \qquad M_2 = -\int_A \sigma_{33}(x_1 - \overline{x}_1)dA$$

No twist means $\quad M_3 = 0$

Moment-Curvature relations:
$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = E\begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix}\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix}$$

Equations of motion: **F=ma:**
$$\frac{dT_i}{dx_3} + p_i = \rho A a_i$$

Angular Momentum:
$$\frac{dM_1}{dx_3} - T_2 - \theta_1 T_3 = 0 \qquad \frac{dM_2}{dx_3} + T_1 - \theta_2 T_3 = 0$$

Boundary Conditions (at ends):

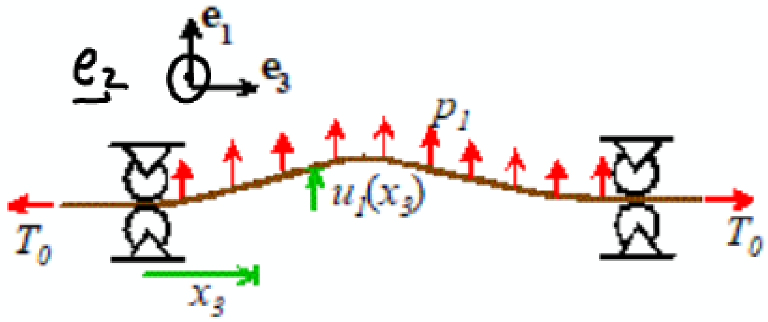
Fixed end: $\mathbf{u} = \mathbf{0}$  Free to move: $\mathbf{T} = \mathbf{0}$

Clamped end: $\boldsymbol{\theta} = \mathbf{0}$  Free to rotate: $\mathbf{M} = \mathbf{0}$

11.2 Simplified beam equations for special cases

① Tensioned cable : Negligible bending resistance $EI/T_3L^2 << 1$

Geometry : $\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \frac{d}{dx_3}\begin{bmatrix}-u_2\\ u_1\end{bmatrix}$



Moment - Curvature relations $[I]=0$

$\Rightarrow \quad M_1 = M_2 = 0$

Angular Momentum $\begin{bmatrix}T_1\\ T_2\end{bmatrix} = T_3\begin{bmatrix}\theta_2\\ -\theta_1\end{bmatrix} = T_3\frac{d}{dx_3}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ (1)

Linear Momentum $\frac{d}{dx_3}\begin{bmatrix}T_1\\ T_2\end{bmatrix} + \begin{bmatrix}p_1\\ p_2\end{bmatrix} = \rho A\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ (2)

Boundary conditions either $\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ given or $\begin{bmatrix}T_1\\ T_2\end{bmatrix}$ given @ ends

Axial eom : $\frac{dT_3}{dx_3} + p_3 = \rho A a_3$

Simplify by combining (1) & (2)

$$\boxed{\frac{d}{dx_3} T_3 \frac{d}{dx_3}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \rho A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}$$

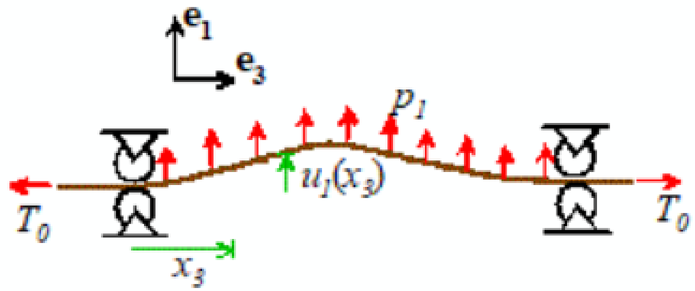
In many cases no distributed axial load $p_3 = 0$
& static equilibrium

$\frac{dT_3}{dx_3} = 0 \Rightarrow T_3 = \text{constant}$

BC @ $x_3 = 0$, $x_3 = L \Rightarrow T_3 = T_0$ (given tension)

$$\boxed{T_0 \frac{d^2}{dx_3^2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \rho A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}}$$

Example: String is subjected to uniform transverse load and has zero displacement at both ends. Find the static equilibrium deflection



Governing eq (T_3 = const case)

$$T_0 \frac{d^2}{dx_3^2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad u_2 = 0 \qquad T_0 \frac{d^2 u_1}{dx_3^2} + p_1 = 0$$

$$\Rightarrow \quad u_1 = -\frac{p_1}{T_0}\frac{x_3^2}{2} + A x_3 + B$$

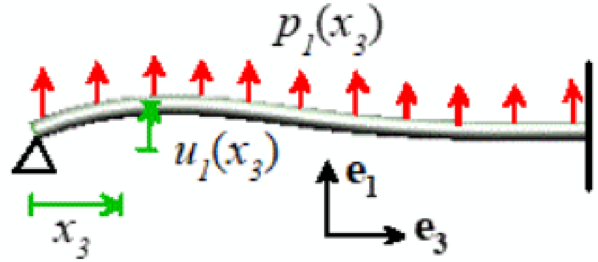
BCs: $u_1 = 0$ @ $x_3 = 0$ $\Rightarrow$ $B = 0$

$u_1 = 0$ @ $x_3 = L$ $\Rightarrow$ $A = \frac{p_1 L}{2T_0}$

$$u_1 = \frac{p_1}{2T_0} x_3 (L - x_3)$$

(parabola ; max deflection $p_1 L^2 / 8T_0$)

② Beam with negligible axial load $\quad EI/T_3L^2 \gg 1$



Geometry
$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \frac{d}{dx_3}\begin{bmatrix}-u_2\\ u_1\end{bmatrix}$$

$$\begin{bmatrix}\kappa_1\\ \kappa_2\end{bmatrix} = \frac{d}{dx_3}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \frac{d^2}{dx_3^2}\begin{bmatrix}-u_2\\ u_1\end{bmatrix}$$

Moment - curvature
$$\begin{bmatrix}M_1\\ M_2\end{bmatrix} = E\begin{bmatrix}I_{11} & -I_{12}\\ -I_{12} & I_{22}\end{bmatrix}\begin{bmatrix}\kappa_1\\ \kappa_2\end{bmatrix} = E\begin{bmatrix}I_{11} & -I_{12}\\ -I_{12} & I_{22}\end{bmatrix}\frac{d^2}{dx_3^2}\begin{bmatrix}-u_2\\ u_1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-M_2\\ M_1\end{bmatrix} = -E\begin{bmatrix}I_{22} & I_{12}\\ I_{12} & I_{11}\end{bmatrix}\frac{d^2}{dx_3^2}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Angular momentum
$$\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \frac{d}{dx_3}\begin{bmatrix}-M_2\\ M_1\end{bmatrix}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = -E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^3}{dx_3^3} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Linear momentum $\qquad \frac{d}{dx_3} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \rho A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

Combine :

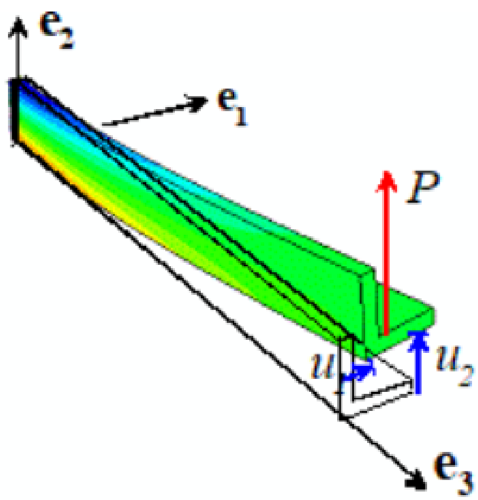
$$\boxed{E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^4}{dx_3^4} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \rho A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

eom for 3D beam

Boundary conditions :

either $\left.\begin{matrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \end{matrix}\right\}$ given $\qquad \left.\begin{matrix} \begin{bmatrix} -M_2 \\ M_1 \end{bmatrix} \\ \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \end{matrix}\right\}$ given $\quad$ @ $x_3 = 0$, $x_3 = L$

page **Example:** Cantilever beam with L shaped x-sect is subjected to transverse load at end. Find the deflection.

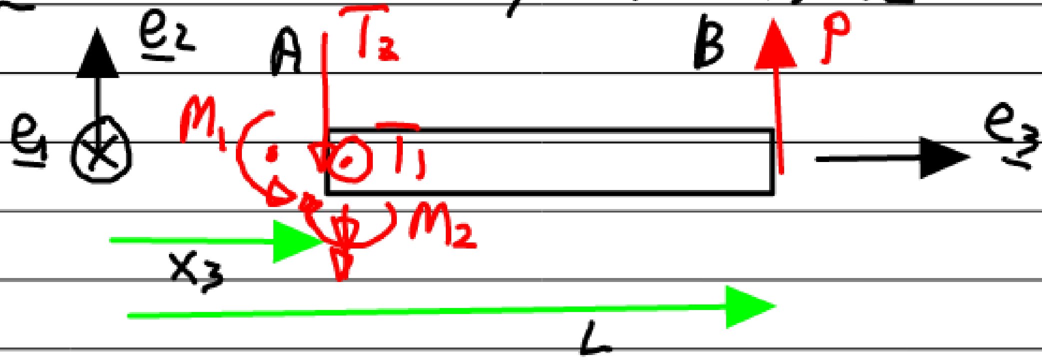


Assume $$\mathbf{I} = \frac{a^3 t}{8}\begin{bmatrix} 5/3 & 1 \\ 1 & 5/3 \end{bmatrix} = \begin{bmatrix} I_{11} & -I_{12} \\ -I_{12} & I_{22} \end{bmatrix}$$

$$I_{11} = I_{22} = \frac{5}{24} a^3 t \qquad I_{12} = -a^3 t/8$$

Try 2 methods : Hand calcs
MATLAB

① Hand Calcs : Use statics to find M_2, M_1
Use $\underline{M}$ - $\underline{\kappa}$ relations to find u_1, u_2



$\Sigma \underline{M} = \underline{0}$ about A

$M_2 = 0 \qquad M_1 + P(L - x_3) = 0$

$$\Rightarrow \begin{bmatrix} -M_2 \\ M_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -P(L-x_3) \end{bmatrix}$$

$$\Rightarrow \cancel{-}E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^2}{dx_3^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \cancel{-}P(L-x_3) \end{bmatrix}$$

$$\Rightarrow \frac{Ea^3t}{24} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \frac{d^2}{dx_3^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ P(L-x_3) \end{bmatrix}$$

Invert $$\frac{d^2}{dx_3^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{24}{Ea^3t} \frac{1}{16} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ P(L-x_3) \end{bmatrix} = \frac{3\,P(L-x_3)}{2Ea^3t} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Integrate $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{3}{2} \frac{P}{Ea^3t} \left(L\frac{x_3^2}{2} - \frac{x_3^3}{6} \right) \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_3 + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

BCs: $$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \frac{d}{dx_3} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad x_3 = 0$$

$$A_1 = A_2 = B_1 = B_2 = 0$$

$$\boxed{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{3}{2} \frac{P}{Ea^3 t} x_3^2 \left(\frac{L}{2} - \frac{x_3}{6} \right) \begin{bmatrix} 3 \\ 5 \end{bmatrix}}$$

MATLAB solution

Solve $E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^4}{dx_3^4} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

with MATLAB "dsolve" with BCs

$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \frac{d}{dx_3} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ @ $x_3 = 0$

$-E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^2}{dx_3^2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (no moment) $x_3 = L$ (3)

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = -E \begin{bmatrix} I_{22} & I_{12} \\ I_{12} & I_{11} \end{bmatrix} \frac{d^3}{dx_3^3} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix} \quad @ \ x_3 = L \qquad (4)$$

Problem : MATLAB can't handle (3) & (4)

Fix : Solve with $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ $\quad \frac{d}{dx_3} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$

$(C_1 - C_4)$ are unknown constants

Use BCs to solve for $(C_1 - C_4)$ & substitute back

```
syms a t EE P L x3 C1 C2 C3 C4 real
syms u1(x3) u2(x3)
II1 = a^3*t/24*[5,3;3,5]; % This is [I11,-I12;-I12;I22] - regular area moment of inertia matrix
II2 = a^3*t/24*[5,-3;-3,5]; % This is [I22,I12;I11,I22] that appears in some of the equations
uvec = [u1(x3);u2(x3)]; %vector of unknown displacements
diffeq = EE*II2*diff(uvec,x3,4) ==[0;0];  % The differential equation
BC1 = subs(uvec,x3,0)==0;  BC2 = subs(diff(uvec,x3),x3,0)==0; % BCs at x3=0 are easy
% Matlab 'dsolve' can't handle the moment/force boundary conditions so instead
% we prescribe the displacement and slope at x3=L using C1,C2,C3,C4 (unknown constants)
% and then solve for these unknown constants using the correct boundary conditions later
BC3 = subs(uvec,x3,L)==[C1;C2];  BC4 = subs(diff(uvec,x3),x3,L)==[C3;C4];
sol = dsolve(diffeq,[BC1,BC2,BC3,BC4]);
uvecsol = [sol.u1;sol.u2] % This solution still contains the unknowns C1,C2,C3,C4
mvec = -EE*II2*diff(uvecsol,x3,2); % NB: mvec contains [-M2;M1]
Tvec = -EE*II2*diff(uvecsol,x3,3); % Tvec contains [T1;T2]
eq1 = subs(mvec,x3,L)==[0;0];  % Moments are zero at x3=L
eq2 = subs(Tvec,x3,L)==[0;P];  % Internal force is [0;P] at x3=L
[C1sol,C2sol,C3sol,C4sol] = solve([eq1,eq2],[C1,C2,C3,C4]); % Solve for C1,C2,C3,C4
uvecsol = simplify(subs(uvecsol,[C1,C2,C3,C4],[C1sol,C2sol,C3sol,C4sol])) % subst back
```

uvecsol =

$$\begin{pmatrix} \frac{x_3^2\,(3\,C_1 - C_3\,L)}{L^2} - \frac{x_3^3\,(2\,C_1 - C_3\,L)}{L^3} \\ \frac{x_3^2\,(3\,C_2 - C_4\,L)}{L^2} - \frac{x_3^3\,(2\,C_2 - C_4\,L)}{L^3} \end{pmatrix}$$

uvecsol =

$$\begin{pmatrix} \frac{3\,P\,x_3^2\,(3\,L - x_3)}{4\,\mathrm{EE}\,a^3\,t} \\ \frac{5\,P\,x_3^2\,(3\,L - x_3)}{4\,\mathrm{EE}\,a^3\,t} \end{pmatrix}$$