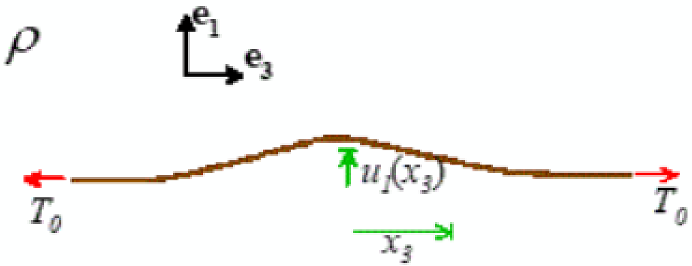
12 Dynamics

Goal: Understand wave propagation, vibrations
- focus only on elastic solids

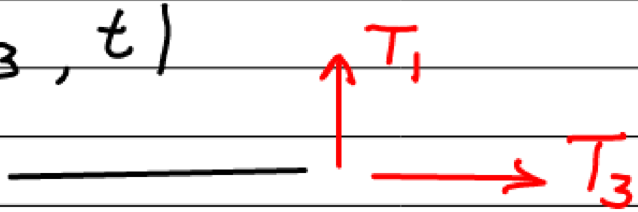
12.1 Travelling waves on infinite stretched string

Example: An infinite string with x-sect area *A* and mass density ρ is stretched by tension T_0
At time *t=0* it is at rest, and has transverse deflection $w_0(x_3)$
Find the subsequent motion



Recall displacement $u_1(x_3, t)$

Internal forces



Governing Equations

$$\frac{\partial T_3}{\partial x_3} + \underbrace{\cancel{p_3}}_{=0} = \rho A \underbrace{\cancel{\frac{\partial^2 u_3}{\partial t^2}}}_{=0} \quad (1)$$

$$\frac{\partial}{\partial x_3}\left\{ T_3 \frac{\partial u_1}{\partial x_3} \right\} + \underbrace{\cancel{p_1}}_{=0} = \rho A \frac{\partial^2 u_1}{\partial t^2} \quad (2)$$

$$T_1 = T_3 \frac{\partial u_1}{\partial x_3}$$

(1) $\Rightarrow \dfrac{\partial T_3}{\partial x_3} = 0 \Rightarrow T_3 = \text{const}$ BCs $\Rightarrow$ constant $= T_0$

Now (2) $\Rightarrow T_0 \dfrac{\partial^2 u_1}{\partial x_3^2} = \rho A \dfrac{\partial^2 u_1}{\partial t^2}$

Hence $\dfrac{\partial^2 u_1}{\partial x_3^2} = \dfrac{1}{c^2} \dfrac{\partial^2 u_1}{\partial t^2}$ $c = \sqrt{\dfrac{T_0}{\rho A}}$ (Wave speed)

1-D wave equation

Now solve wave equation with given initial condition

Guess & check : Try $u_1 = f(x_3 - ct) + g(x_3 + ct)$

[Notation : for $f(\lambda)$ Let $\frac{df}{d\lambda} = f'(\lambda)$ $\frac{d^2f}{d\lambda^2} = f''(\lambda)$]

Subst in wave eq : $f'' + g'' = \frac{1}{c^2}\{c^2 f'' + c^2 g''\}$ ✓ eom satisfied

We find f, g from initial conditions

$$u_1(x_3, 0) = f(x_3) + g(x_3) = W_0(x_3) \qquad (3)$$

$$\left.\frac{\partial u_1}{\partial t}\right|_{t=0} = -\cancel{c}f'(x_3) + \cancel{c}g'(x_3) = 0$$

Integrate $-f + g = A$ (const of integration) (4)

Solve (3), (4) for f, g

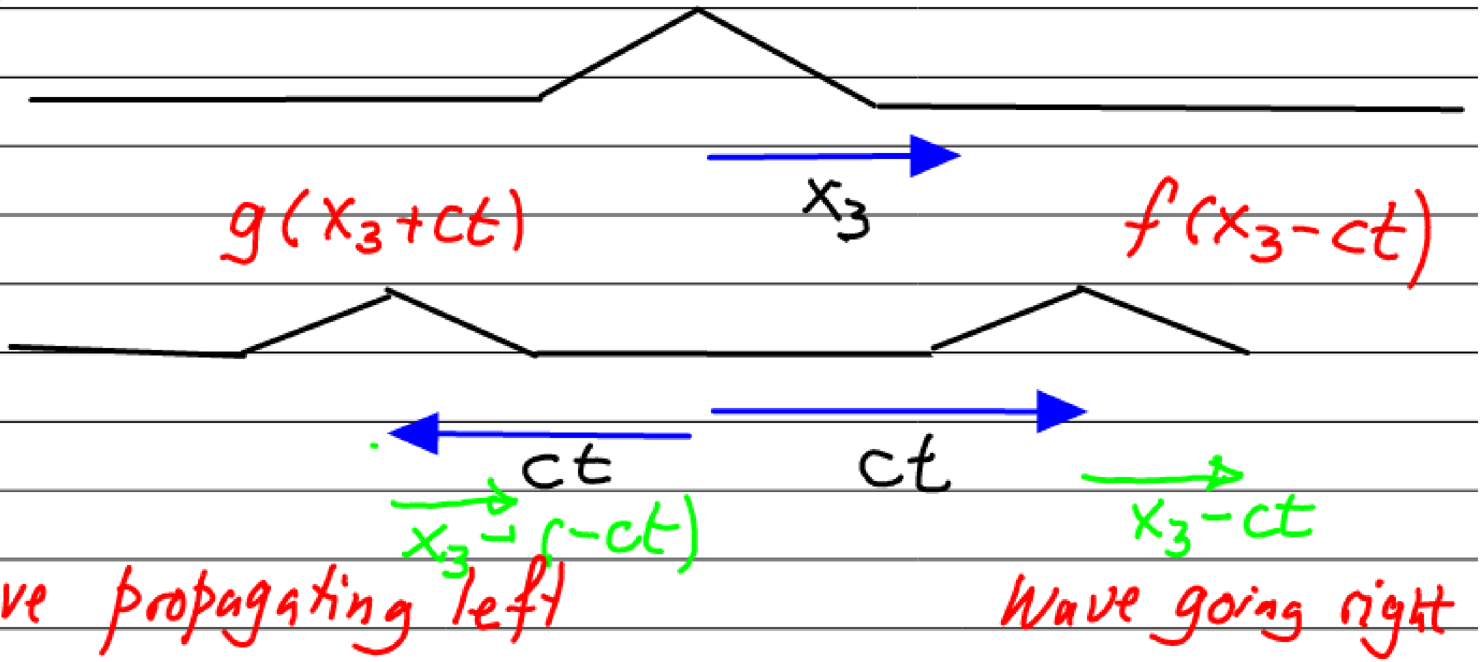
$$g(x_3) = \frac{1}{2}\left(W_0(x_3) + A\right)$$

$$f(x_3) = \frac{1}{2}\left(W_0(x_3) - A\right)$$

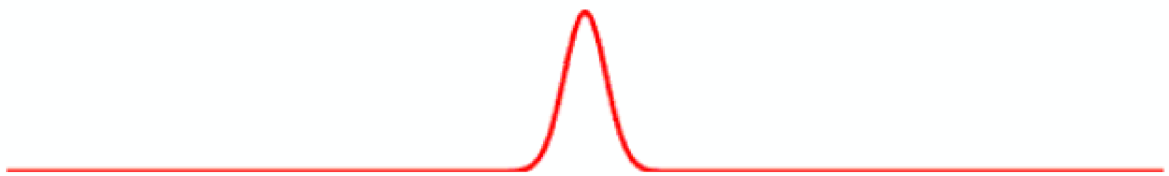
Solution $u_1(x_3, t) = \frac{1}{2} W_0(x_3 - ct) + \frac{1}{2} W_0(x_3 + ct)$

Visualize solution

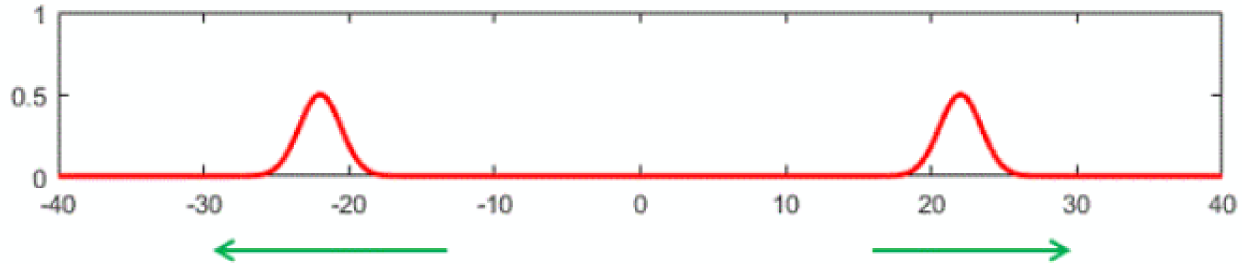
$t = 0$



Solution to wave equation



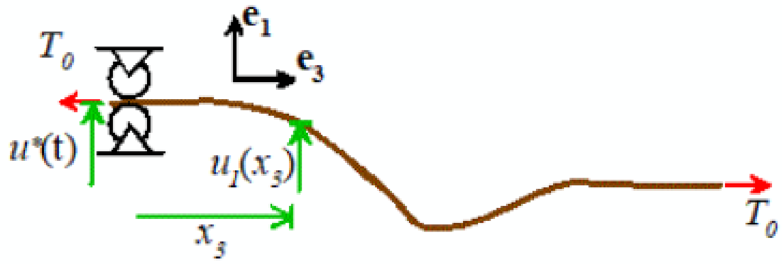
Initial condition $$w_0 = \exp(-x_3^2 / 4)$$



$$g(t, x_3) = \frac{1}{2}\exp(-(x_3 + ct)^2 / 4) \qquad f(t, x_3) = \frac{1}{2}\exp(-(x_3 - ct)^2 / 4)$$

Travelling waves, propagate at speed *c*

Example: An semi- infinite string with x-sect area A and mass density ρ is stretched by tension T_0
At time $t=0$ it is at rest, with zero transverse deflection



The end at $x_3=0$ has prescribed displacement $u^*(t)$

Find the subsequent motion

eg $u^*(t) = A\sin\omega t$

Same EOM (wave eq) same sol $u_1 = f(x_3 - ct) + g(x_3 + ct)$

Find $f(\lambda)$, $g(\lambda)$ using initial conditions & boundary condition

Initial Condition $u_1 = f(x_3) + g(x_3) = 0 \qquad x_3 \geq 0$

$$\frac{\partial u_1}{\partial t} = -cf'(x_3) + cg'(x_3) = 0 \qquad x_3 \geq 0$$

Solve $f(x_3) = A \qquad x_3 \geq 0$

$g(x_3) = -A \qquad x_3 \geq 0$ ~ g is known

We also need $f(\lambda)$ for $\lambda \leq 0$

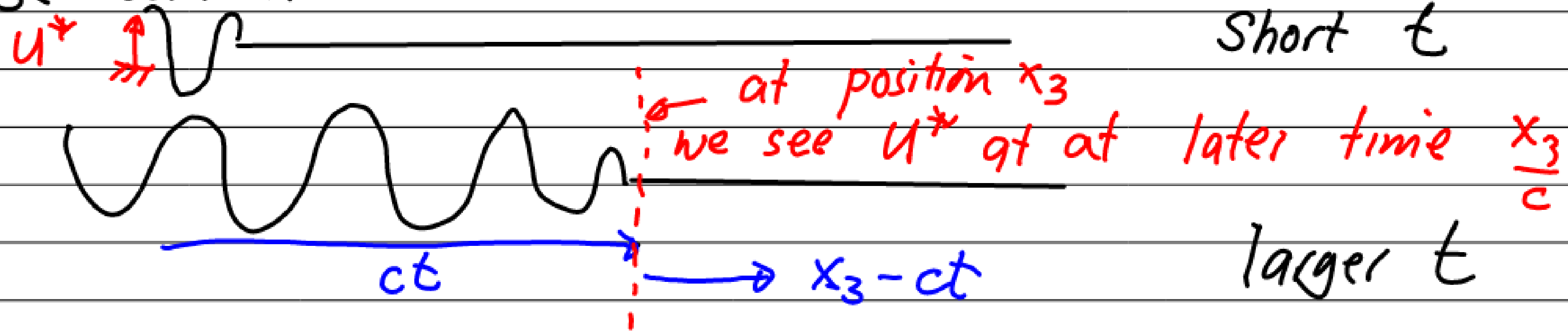
Use BC: $u_1(0,t) = f(-ct) + g(ct) = u^*(t) \quad t \geq 0$

Hence $f(\lambda) = u^*(-t/c) + A \qquad \lambda \leq 0$

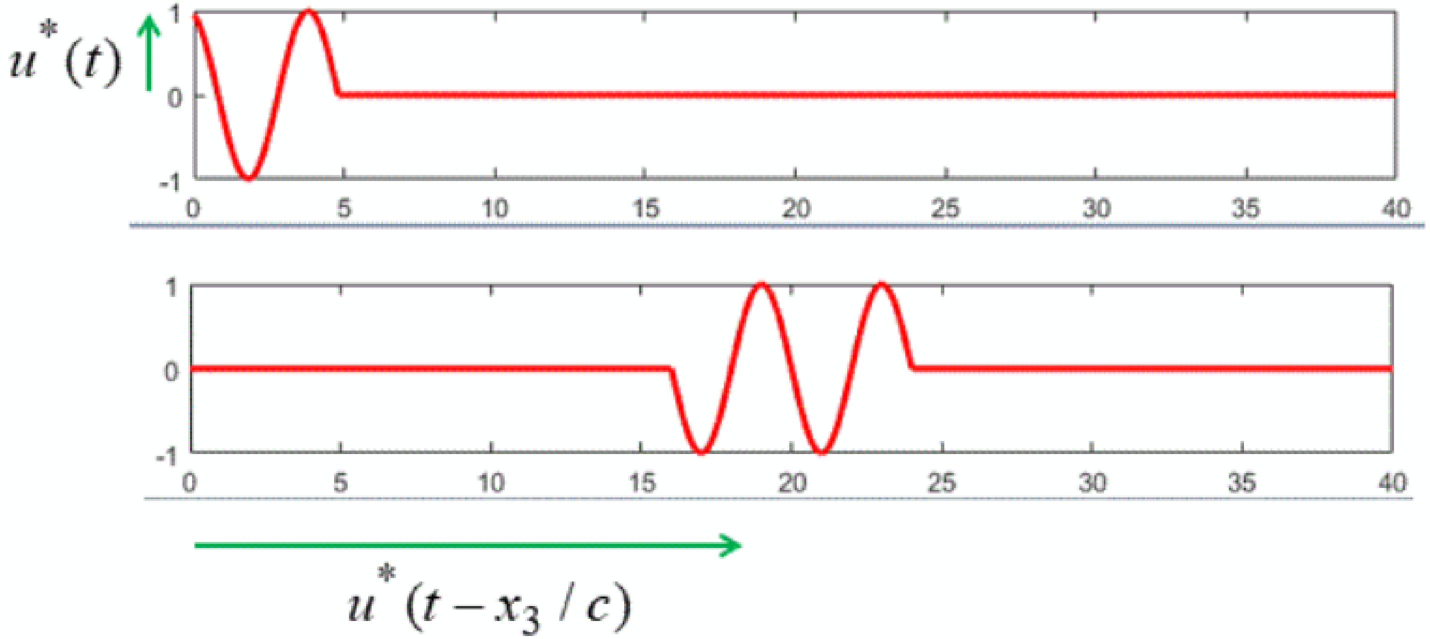
Finally

$$u_1(x_3,t) = f(x_3 - ct) + g(x_3 + ct) = \begin{cases} 0 & x_3 - ct \geq 0 \\ u^*\left(t - \frac{x_3}{c}\right) & x_3 - ct \leq 0 \end{cases}$$

Visualize solution

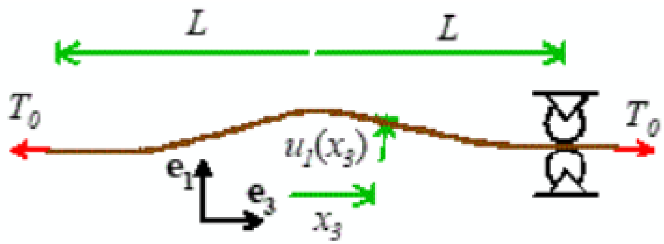


Semi-infinite string forced at one end



Wave arrives at x_3 a time x_3 / c later

Example: A string with length *2L* x-sect area *A* and mass density is stretched by tension T_0
At time *t=0* it is at rest, and has transverse deflection $w_0(x_3)$
It is fixed at $x_3 = L$ and free at $x_3 = -L$
Find the subsequent motion



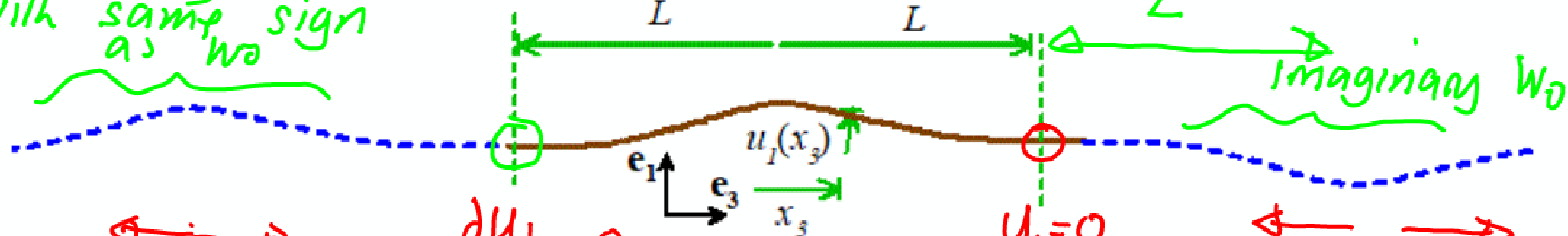
Same eom, same solution $u_1(x_3,t) = f(x_3 - ct) + g(x_3 + ct)$
Find f, g using initial condition & BCs

BCs : $u_1(L,t) = 0 \qquad t \geq 0$

$$T_1 = T_0 \left.\frac{\partial u_1}{\partial x_3}\right|_{x_3=-L} \Rightarrow \left.\frac{\partial u_1}{\partial x_3}\right|_{x_3=-L} \qquad t \geq 0$$

Solve by superposition:
(1) Extend string to $-\infty < x_3 < \infty$
(2) Find initial conditions on $|x_3| > L$ to satisfy BCs

Imaginary W_0 with same sign as W_0



splits into two waves

$\frac{\partial u_1}{\partial x_3} = 0$

splits into 2 propagating waves

$u_1 = 0$

splits into two waves opposite to W_0

$$u_1(x_3, t) = \frac{1}{2} W_0 (x_3 - ct) + \frac{1}{2} W_0 (x_3 + ct)$$

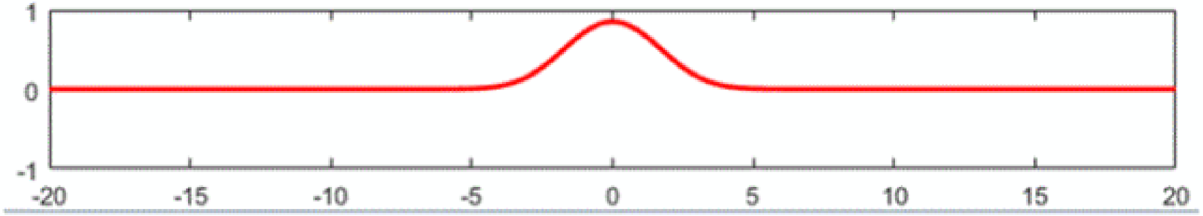
1st reflection $$- \frac{1}{2} W_0 (x_3 - 2L + ct) + \frac{1}{2} W_0 (x_3 + 2L - ct)$$

2nd reflection $$- \frac{1}{2} W_0 (x_3 + 4L - ct) - \frac{1}{2} W_0 (x_3 - 4L + ct)$$

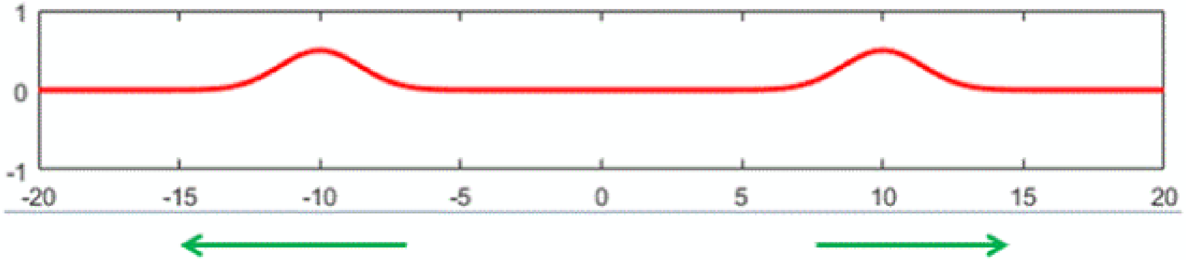
3rd reflection $$+ \frac{1}{2} W_0 (x_3 - 6L + ct) - \frac{1}{2} W_0 (x_3 + 6L - ct)$$

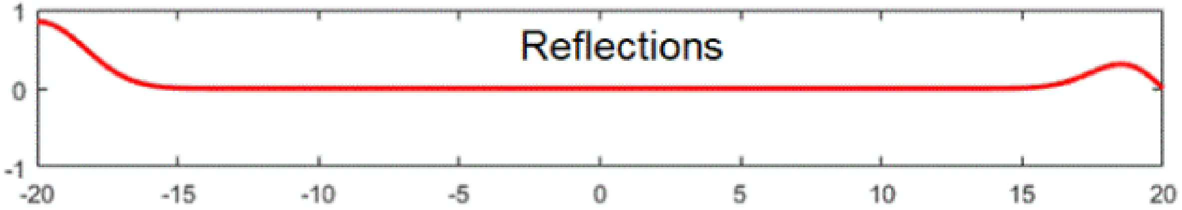
Continues

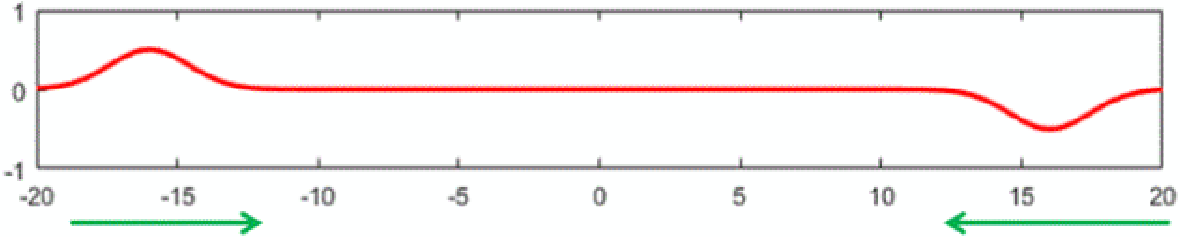
Reflections at free and fixed ends



Initial condition $u(0, x_3) = \exp(-x_3^2 / 4)$







At a fixed end:

- Positive displacement reflects as negative displacement
- Positive transverse force reflects as positive transverse force

At a free end:

- Positive displacement reflects as positive displacement
- Positive transverse force reflects as negative transverse force