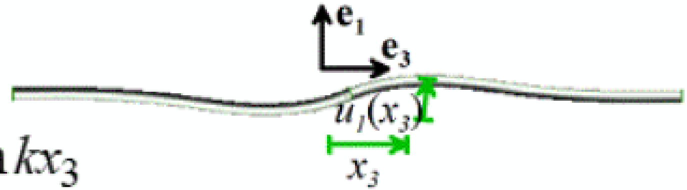
12.2 Travelling waves in beams

Example: An infinite beam with x-sect area *A* and mass density ρ and moment of inertia $I_{22} = I \quad I_{12} = 0$
At time *t=0* it is at rest, and has transverse deflection $w_0(x_3) = B\sin kx_3$
Find the subsequent motion



Note $k = 2\pi/\lambda \quad \lambda$ = wavelength

Governing equation $EI \dfrac{\partial^4 u_1}{\partial x_3^4} + \rho A \dfrac{\partial^2 u_1}{\partial t^2} = 0$

Note $u_1 = f(x_3 \pm ct)$ won't satisfy eom in general

Try harmonic solution

$$u_1(x_3, t) = U_0^+ \sin k(x_3 - ct) + U^- \sin k(x_3 + ct)$$

Substitute into eom:

$$(EI k^4 - \rho A c^2) \left[U_0^+ \sin k(x_3 - ct) + U_0^- \sin k(x_3 + ct) \right] = 0$$

Can satisfy eom with $\boxed{c = \sqrt{\frac{EI}{\rho A}}\, k^2 = \sqrt{\frac{EI}{\rho A}} \left(\frac{2\pi}{\lambda}\right)^2}$

Wave speed depends on wavelength

"Dispersive" wave

Relation between c & k called "dispersion relation"

Choose U_0^+, U_0^- to satisfy initial conditions

$$u_1(x_3, 0) = (U_0^+ + U_0^-) \sin k x_3 = B \sin k x_3$$

$$\left.\frac{\partial u_1}{\partial t}\right|_{t=0} = c(-U_0^+ + U_0^-) \cos(k x_3) = 0$$

Hence $u_0^+ = u_0^- = B/2$

$$\boxed{u_1(x_3, t) = \frac{B}{2}\left(\sin k(x_3 - ct) + \sin k(x_3 + ct) \right)}$$

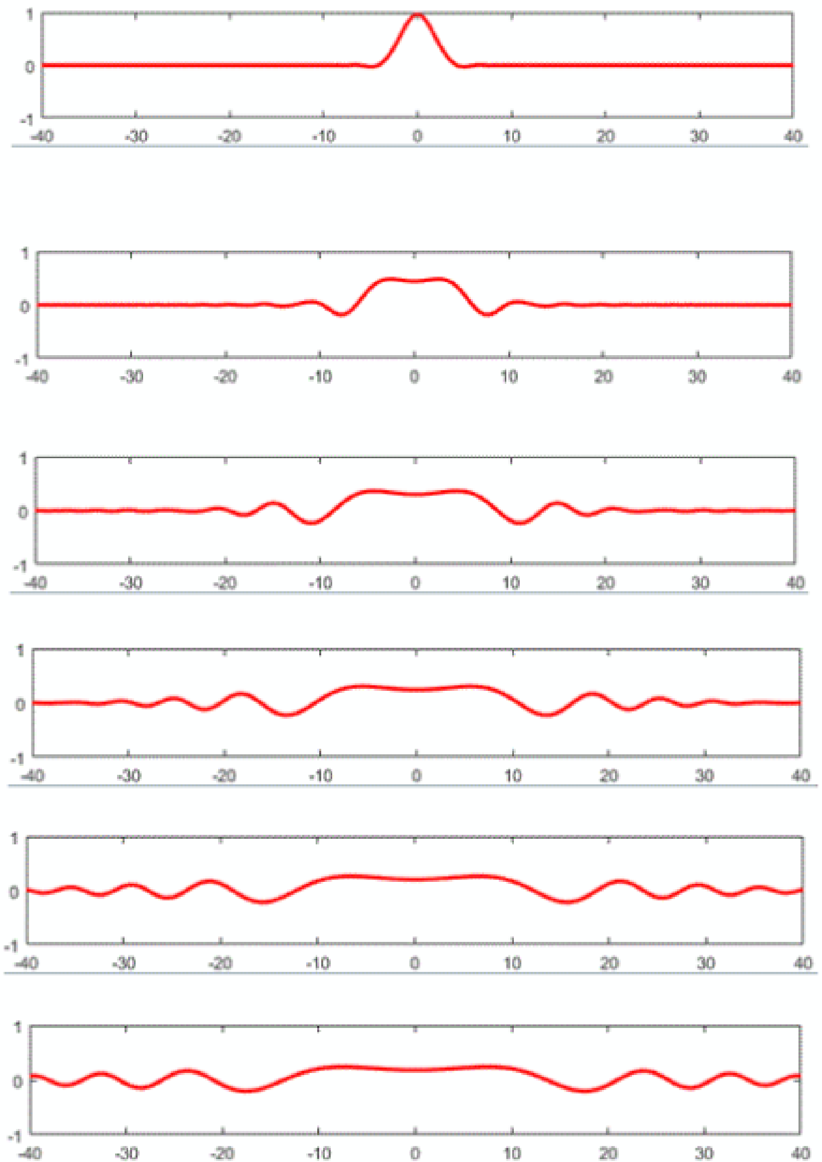
Harmonic initial disturbance generates two travelling waves (same as string) but c depends on λ

For general initial disturbance $w_0(x_3)$ we could express w_0 as a fourier series or spectrum – contains many wavelengths

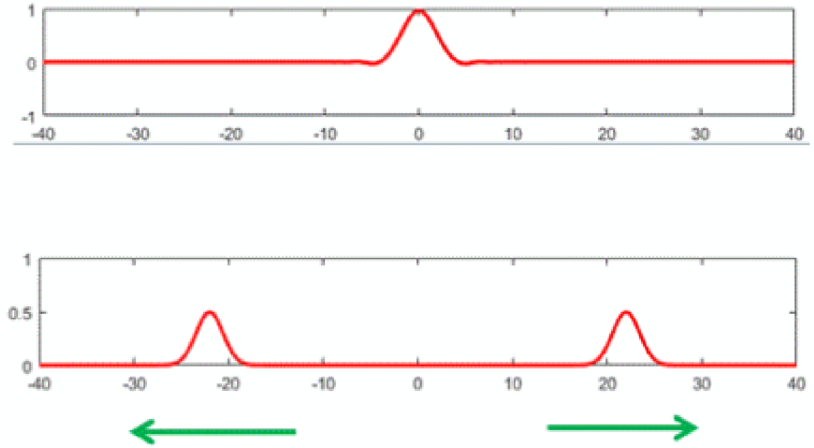
Each wavelength propagates @ different c

Short wavelengths propagate quickly

Beam - dispersive



String - non dispersive



12.3 Plane Waves in an infinite solid

Consider large elastic solid, props E, ν, ρ (isotropic)

What wave motion satisfies eom?

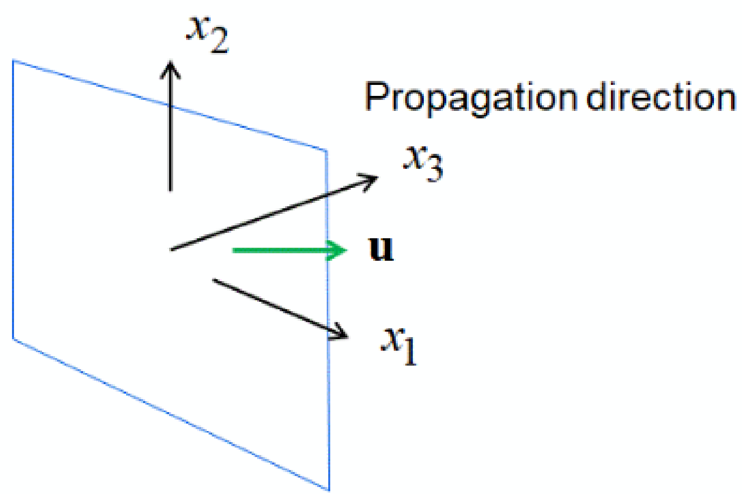
EOM:

$$\varepsilon = (\nabla \underline{u} + (\nabla \underline{u})^T)/2$$

$$\underline{\sigma} = [C]\,\underline{\varepsilon}$$

$$\nabla \cdot \sigma = \rho\, \partial^2 \underline{u} / \partial t^2$$

Consider "plane wave" solution propagating in x_3 direction



$$\underline{u}(\underline{x},t) = u_1(x_3 \pm ct)\underline{e}_1 + u_2(x_3 \pm ct)\underline{e}_2 + u_3(x_3 \pm ct)\underline{e}_3$$

(Note $\underline{u}$ independent of x_1, x_2)

Strains $\underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{23}]$

$$= \left[0, 0, \frac{\partial u_3}{\partial x_3}, 0, \frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3}\right]$$

Stresses $\underline{\sigma} = [\sigma_{11}, \sigma_{22} \ldots \sigma_{12}, \sigma_{13}, \sigma_{23}] = [C]\underline{\varepsilon}$

$$= \frac{E}{(1+\nu)}\left[\frac{\nu}{1-2\nu}\frac{\partial u_3}{\partial x_3}, \frac{\nu}{1-2\nu}\frac{\partial u_3}{\partial x_3}, \frac{(1-\nu)}{(1-2\nu)}\frac{\partial u_3}{\partial x_3}, 0, \frac{1}{2}\frac{\partial u_1}{\partial x_3}, \frac{1}{2}\frac{\partial u_2}{\partial x_3}\right]$$

Linear momentum

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho\frac{\partial^2 u_1}{\partial t^2} \Rightarrow \frac{E}{2(1+\nu)}\frac{\partial^2 u_1}{\partial x_3^2} = \rho\frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2} = \frac{E}{2(1+\nu)} \frac{\partial^2 u_2}{\partial x_3^2} = \rho \frac{\partial^2 u_2}{\partial t^2}$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_3} + \frac{\partial \sigma_{33}}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2} = \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} \frac{\partial^2 u_3}{\partial x_3^2} = \rho \frac{\partial^2 u_3}{\partial t^2}$$

Two wave equations

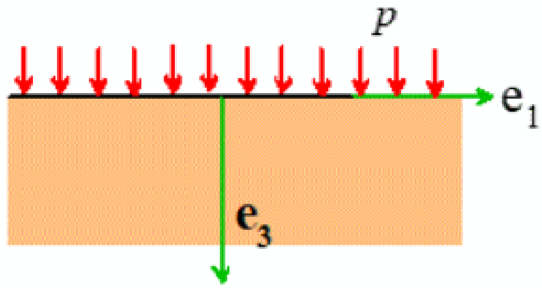
$$\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c_s^2} \frac{\partial^2 u_1}{\partial t^2}$$

$$\frac{\partial^2 u_3}{\partial x_3^2} = \frac{1}{C_L^2} \frac{\partial^2 u_3}{\partial t^2}$$

$C_s = \sqrt{\dfrac{E}{2(1+\nu)\rho}}$ → shear wave speed or "S" wave speed

$C_L = \sqrt{\dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)}}$ → pressure or "P" wave

Example: A large elastic solid is at rest and stress free for *t<0*. For time *t>0* its surface is subjected to a constant uniform pressure *p*. Calculate the stress and velocity distribution in the solid.



Recall wave speed $c_L = \sqrt{\dfrac{E(1-\nu)}{(1-2\nu)(1+\nu)\rho}}$

Try a p wave solution $u_3(x_3, t) = f(x_3 - c_L t)$

Automatically satisfy EOM

Note $\dfrac{\partial u_3}{\partial t} = -c_L f'(x_3 - c_L t)$ $\qquad f' = \dfrac{\partial f(\lambda)}{\partial \lambda}$

$\sigma_{33} = \dfrac{E(1-\nu)}{(1+\nu)(1-2\nu)} \dfrac{\partial u_3}{\partial x_3} = c_L^2 \rho f'(x_3 - c_L t)$ $\left[\sigma_{11} = \sigma_{22} = \dfrac{\nu}{1-\nu} \sigma_{33} \right.$

all other $\sigma_{ij} = 0 \Big]$

Find f using initial conditions & BC

Initial condition

$$u_3(x_3, 0) = f(x_3) = 0 \qquad x_3 > 0$$

$$\frac{\partial u_3}{\partial t} = -c_L f'(x_3) = 0 \qquad x_3 > 0$$

Also need $f(\lambda)$ for $\lambda < 0$

Boundary condition $\quad \underline{n}\,\sigma = \underline{t} \quad$ on $x_3 = 0$

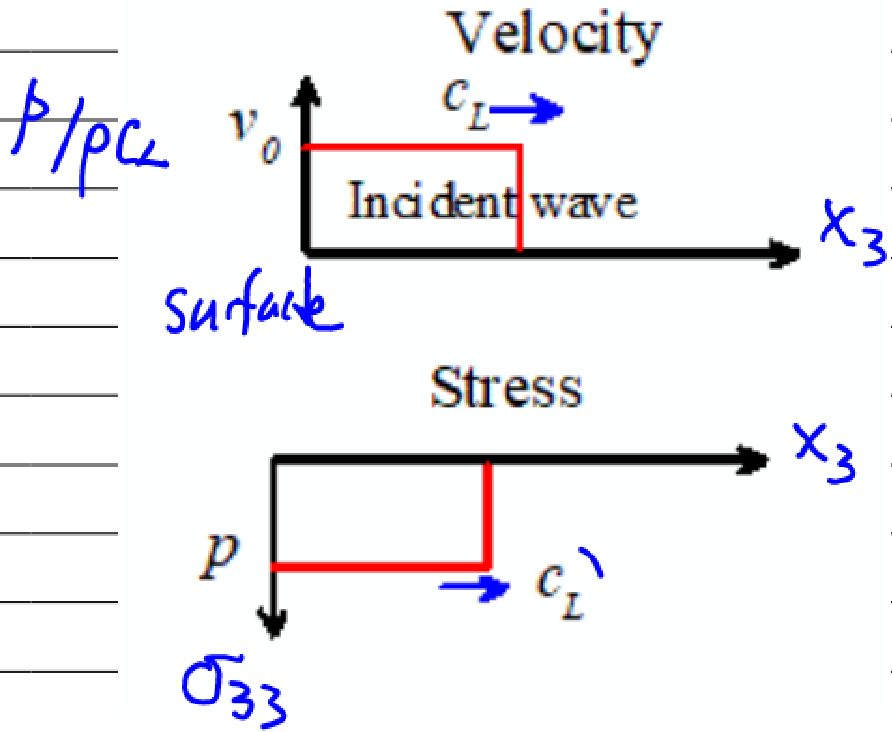
$\underline{n} = -\underline{e}_3 \qquad \underline{t} = p\,\underline{e}_3 \quad$ (Given)

$$\Rightarrow -\sigma_{33} = p \quad \Rightarrow c_L^2 \rho f'(-c_L t) = -p \qquad t > 0$$

$$\Rightarrow f'(\lambda) = \frac{-p}{\rho c_L^2} \quad \Rightarrow f(\lambda) = \frac{-p}{\rho c_L^2}\lambda + \text{const}$$

We know $f'(0) = 0 \Rightarrow$ const $= 0$

$$\boxed{\begin{aligned}
u_3(x_3, t) &= \begin{cases} 0 & x_3 - c_L t > 0 \\ -\dfrac{p}{\rho c_L^2}(x_3 - c_L t) & x_3 - c_L t < 0 \end{cases} \\
v_3 = \frac{\partial u_3}{\partial t} &= \begin{cases} 0 & x_3 - c_L t > 0 \\ \dfrac{p}{\rho c_L} & x_3 - c_L t < 0 \end{cases} \\
\sigma_{33} = \rho c_L^2 \frac{\partial u_3}{\partial x_3} &= \begin{cases} 0 & x_3 - c_L t > 0 \\ -p & x_3 - c_L t < 0 \end{cases}
\end{aligned}}$$


Velocity
c_L
$p/\rho c_L$
v_0
Incident wave
x_3
surface
Stress
x_3
p
c_L
σ_{33}