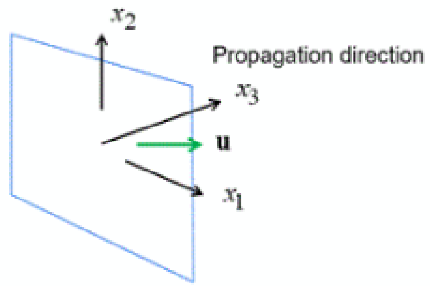
Review

Plane waves in large elastic solids:

1. Deformation $\mathbf{u}(x_3,t) = u_1(x_3,t)\mathbf{e}_1 + u_2(x_3,t)\mathbf{e}_2 + u_3(x_3,t)\mathbf{e}_3$
2. Governing equations

$$\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c_s^2}\frac{\partial^2 u_1}{\partial t^2} \qquad \frac{\partial^2 u_2}{\partial x_3^2} = \frac{1}{c_s^2}\frac{\partial^2 u_2}{\partial t^2} \qquad \frac{\partial^2 u_3}{\partial x_3^2} = \frac{1}{c_L^2}\frac{\partial^2 u_3}{\partial t^2}$$

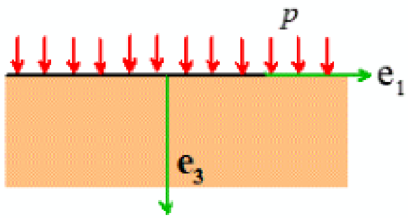
$$\sigma_{13} = \sigma_{31} = \frac{E}{2(1+\nu)}\frac{\partial u_1}{\partial x_3} = \rho c_s^2 \frac{\partial u_1}{\partial x_3} \qquad \sigma_{33} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial u_3}{\partial x_3} = \rho c_L^2 \frac{\partial u_3}{\partial x_3}$$



3. Wave speeds

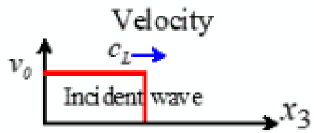
$$c_s = \sqrt{\frac{E}{2(1+\nu)\rho}} \quad \text{Shear (S) wave} \qquad c_L = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}} \quad \text{Pressure (P) wave}$$

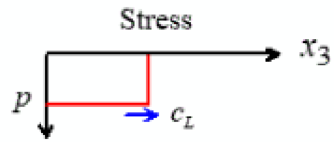
Example: A large elastic solid is at rest for *t<0*. For time *t>0* its surface is subjected to a constant uniform pressure *p*. Calculate the stress and velocity distribution in the solid.



Solution is a P wave $\quad u_3 = \dfrac{p}{\rho c_L}\langle t - x_3/c_L \rangle \qquad \langle x \rangle = \begin{cases} x & x>0 \\ 0 & x<0 \end{cases}$

$$v_3 = \frac{\partial u_3}{\partial t} = \frac{p}{\rho c_L} H(t - x_3/c_L) \qquad \sigma_{33} = -pH(t - x_3/c_L) \qquad H(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$



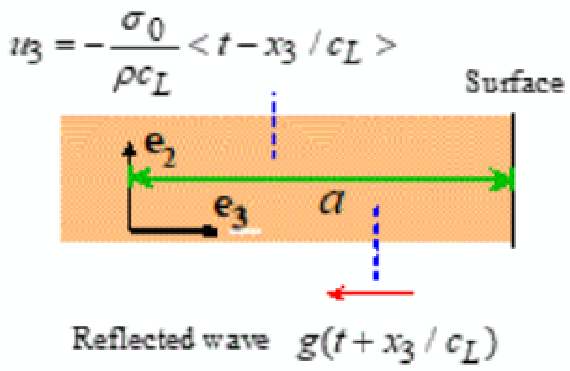


(S wave would be generated by shear stress on surface)

Example: A P wave with displacement, velocity, and stress

$$u_3 = -\frac{\sigma_0}{\rho c_L} < t - x_3 / c_L > \qquad v_3 = -\frac{\sigma_0}{\rho c_L} H(t - x_3 / c_L) \qquad \sigma_{33} = \sigma_0 H(t - x_3 / c_L)$$

reflects off a free surface. Calculate the stress and velocity distribution in the solid after the reflection.



Notation $\quad < x > = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \qquad H(x) = \frac{d<x>}{dx}$$

Solve eom by physical intuition: we expect reflection to be a plane wave propagating along $-x_3$ direction

$$u(x_3, t) = -\frac{\sigma_0}{\rho c_L} < t - x_3 / c_L > + g(t + x_3 / c_L)$$

[Automatically satisfy wave eq; work out g from boundary condition]

From earlier recall $\sigma_{33} = \rho c_L^2 \dfrac{\partial u_3}{\partial x_3}$

$$\Rightarrow \sigma_{33} = \sigma_0 H(t - x_3/c_L) + \rho c_L g'(t + x_3/c_L)$$

$$\sigma_{11} = \sigma_{22} = \nu \sigma_{33}/(1-\nu) \qquad \text{all other } \sigma_{ij} = 0$$

Boundary condition: Surface traction free $\underline{n}\,\sigma = \underline{0}$

$$\underline{n} = \underline{e}_3 = [0\;0\;1] \qquad \sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

$$\Rightarrow \sigma_{33} = 0 \quad \text{on} \quad x_3 = a$$

Hence $\sigma_0 H(t - a/c_L) + \rho c_L g'(t + a/c_L) = 0$

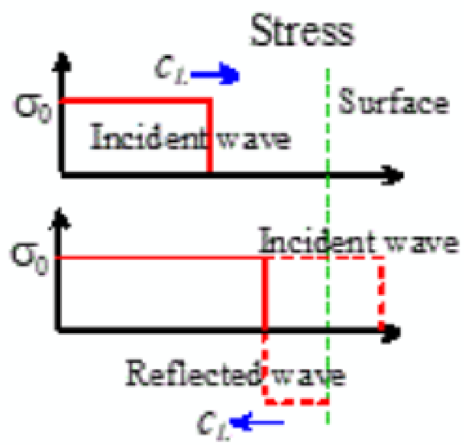
$$g'(t) = -\frac{\sigma_0}{\rho c_L} H(t - 2a/c_L)$$

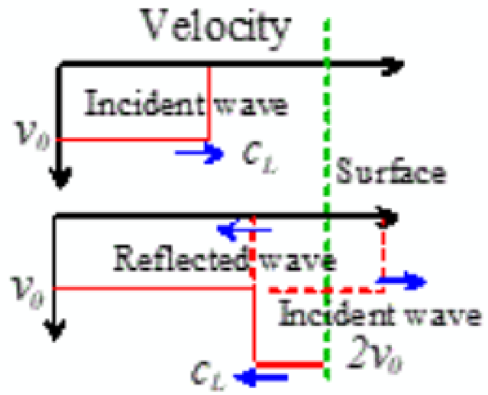
Integrate

$$g(t) = -\frac{\sigma_0}{\rho c_L} \langle t - 2a/c_L \rangle$$

Finally

$$\boxed{\begin{aligned} u_3(x_3,t) &= -\frac{\sigma_0}{\rho c_L}\langle t - x_3/c_L\rangle - \frac{\sigma_0}{\rho c_L}\langle t+(x_3-2a)/c_L\rangle \\ v = \frac{\partial u_3}{\partial t} &= -\frac{\sigma_0}{\rho c_L} H(t - x_3/c_L) - \frac{\sigma_0}{\rho c_L} H(t+(x_3-2a)/c_L) \\ \sigma_{33} = \rho c_L^2 \frac{\partial u_3}{\partial x_3} &= \sigma_0 H(t - x_3/c_L) - \sigma_0 H(t+(x_3-2a)/c_L) \end{aligned}}$$





Summary:
Reflected wave has:
- same velocity as incident wave
- Opposite stress to incident wave

12.3 Other wave speeds in elastic solids

1D Bar (approximate)

Assume uniaxial stress $\sigma_{11} = \sigma$

all other $\sigma_{ij} = 0$

Hence $\varepsilon_{11} = \partial u_1 / \partial x_1 = \sigma / E$

Linear momentum $\dfrac{\partial \sigma}{\partial x_1} = \rho \dfrac{\partial^2 u_1}{\partial t^2} \Rightarrow E \dfrac{\partial^2 u_1}{\partial x_1^2} = \rho \dfrac{\partial^2 u_1}{\partial t^2}$

$$\boxed{\Rightarrow \frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_B^2} \frac{\partial^2 u_1}{\partial t^2} \qquad c_B = \sqrt{\frac{E}{\rho}} \quad \text{(wave speed)}}$$

(approximate solution because linear momentum not satisfied in $\underline{e}_2$, $\underline{e}_3$ directions) $\square$

Rayleigh wave

Special wave that propagates at the surface of an elastic solid

Wave speed satisfies
$$\left(2-\frac{c_R^2}{c_s^2}\right)^2-4\left(1-\frac{(1-2\nu)}{2(1-\nu)}\frac{c_R^2}{c_s^2}\right)^{1/2}\left(1-\frac{c_R^2}{c_s^2}\right)^{1/2}=0$$

Displacement

$$u_1=\frac{U_0 ik}{(k^2-\beta_T^2)\beta_L}\exp(ik(x_1-c_R t))\left\{(k^2+\beta_T^2)\exp(-\beta_L x_2)-2\beta_L\beta_T\exp(-\beta_T x_2)\right\}$$

$$u_2=\frac{U_0}{(k^2-\beta_T^2)}\exp(ik(x_1-c_R t))\left\{2k^2\exp(-\beta_T x_2)-(k^2+\beta_T^2)\exp(-\beta_L x_2)\right\}$$

$$\beta_L=k\sqrt{1-c_R^2/c_L^2} \qquad \beta_T=k\sqrt{1-c_R^2/c_s^2}$$

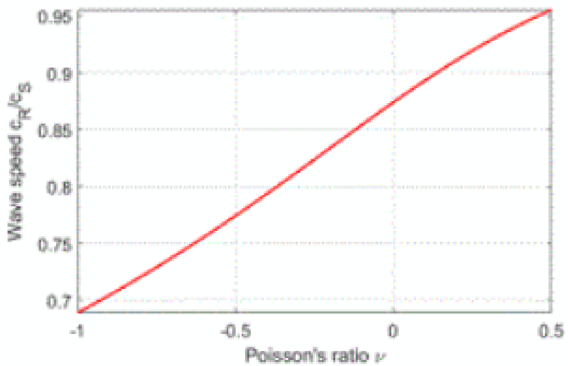
Wave number $k=2\pi/\lambda$

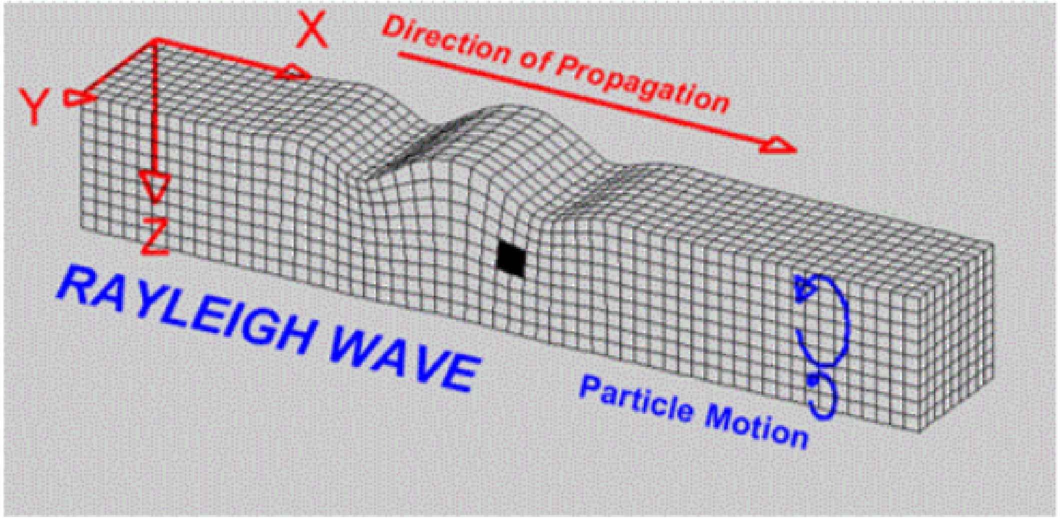
Stress

$$\sigma_{11}=\frac{U_0 E\exp(ik(x_1-c_R t))}{(k^2-\beta_T^2)(1+\nu)(1-2\nu)\beta_L}\left\{k^2\left[\nu(\beta_L^2+\beta_T^2)-(1-\nu)(k^2+\beta_T^2)\right]\exp(-\beta_L x_2)\right.$$
$$\left.+2k^2\beta_T\beta_L(1-2\nu)\exp(-\beta_T x_2)\right\}$$

$$\sigma_{22}=\frac{U_0 E\exp(ik(x_1-c_R t))}{(k^2-\beta_T^2)(1+\nu)(1-2\nu)\beta_L}\left\{(k^2+\beta_T^2)\left[(1-\nu)\beta_L^2-\nu k^2\right]\exp(-\beta_L x_2)\right.$$
$$\left.-2k^2\beta_T\beta_L(1-2\nu)\exp(-\beta_T x_2)\right\}$$

$$\sigma_{12}=\frac{iU_0 kE(k^2+\beta_T^2)}{(k^2-\beta_T^2)(1+\nu)}\exp(ik(x_1-c_R t))\left\{\exp(-\beta_T x_2)-\exp(-\beta_L x_2)\right\}$$

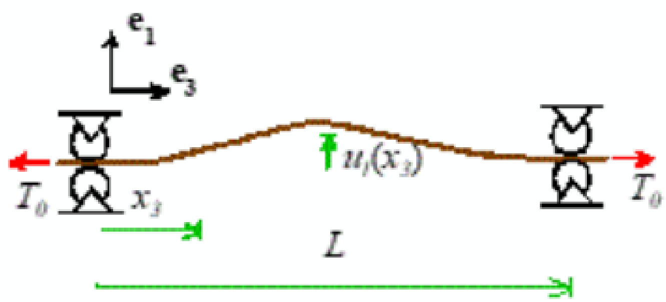




12.4 Vibrations in elastic solids

Goal: Find natural frequencies of vibration
Illustrate method with examples

Example: A string with length *L*, x-sect area *A* and density ρ under axial tension T_0 is pinned at both ends. Find its natural frequencies and mode shapes



① Governing eq:

$$\frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} \qquad c = \sqrt{\frac{T_0}{\rho A}}$$

② Look for harmonic solutions in time

$$u_1 = \underbrace{f_n(x_3)}_{\text{mode shape}} \exp(\pm i \omega_n t)$$

natural frequency

Governing eq: $\frac{d^2 f_n}{dx_3^2} \exp(\pm i\omega_n t) = -\frac{\omega^2}{c^2} f_n \exp(\pm i\omega t)$

$$\Rightarrow \frac{d^2 f_n}{dx_3^2} + \frac{\omega^2}{c^2} f_n = 0 \qquad (1)$$

Solution $f_n = A_n \sin k_n x_3 + B_n \cos k_n x_3$

$$(1) \Rightarrow \left(-k_n^2 + \frac{\omega_n^2}{c^2}\right) f_n = 0 \Rightarrow k_n^2 = \frac{\omega_n^2}{c^2}$$

$$\omega_n = c k_n$$

③ Find k_n from boundary conditions

$u_1 = 0 \qquad x_3 = 0$

$u_1 = 0 \qquad x_3 = L$

Write these in matrix form

$$\begin{array}{l} u_1 = 0 \quad x_3 = 0 \rightarrow \\ u_1 = 0 \quad x_3 = L \rightarrow \end{array} \underbrace{\begin{bmatrix} 0 & 1 \\ \sin k_n L & \cos k_n L \end{bmatrix}}_{H} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2)$$

For nonzero A_n, B_n $\det(H) = 0$

$$\Rightarrow -\sin k_n L = 0 \qquad \Rightarrow k_n L = n\pi$$

Finally $\omega_n = c\, k_n = \dfrac{n\pi}{L}\sqrt{\dfrac{T_0}{\rho A}}$

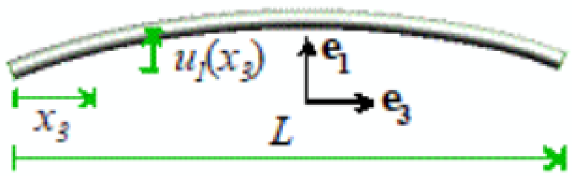
k_n

Can find mode shapes by substituting into (2) & finding A_n, B_n that satisfy BCs

Here $B_n = 0 \Rightarrow$ mode shape $f_n = \sin k_n x_3$

deformable solids have ∞ # nat freqs

Example: A beam with modulus E, inertia I, length L, x-sect area A and density ρ has free ends. Calculate its natural frequencies



① Governing eq: $EI \dfrac{\partial^4 u_1}{\partial x_3^4} + \rho A \dfrac{\partial^2 u_1}{\partial t^2} = 0$

② Look for harmonic sols as before : subst into eom

$$(1) \qquad EI \frac{d^4 f_n}{dx_3^4} - \omega_n^2 \rho A f_n = 0$$

Solution $A_n \sin k_n x_3 + B_n \cos k_n x_3 + C_n \overbrace{\sinh(k_n x_3)}^{(e^x - e^{-x})/2} + D_n \overbrace{\cosh(k_n x_3)}^{(e^x + e^{-x})/2}$

Subst in (1) : $\left(k_n^4 - \omega_n^2 \dfrac{\rho A}{EI}\right) f_n = 0$

$$\omega_n = k_n^2 \sqrt{\frac{EI}{\rho A}}$$

③ Find k_n from boundary conditions

Ends of beam are free: No bending moment, No transverse force $\Big\}$ $x_3 = 0$, $x_3 = L$

Recall $M_2 = EI \dfrac{\partial^2 u_1}{\partial x_3^2}$ $\qquad T_1 = -EI \dfrac{\partial^3 u_1}{\partial x_3^3}$

Express BCs in matrix form

$$
\begin{array}{ll}
M=0 & x_3=0 \rightarrow \\
T=0 & x_3=0 \rightarrow \\
M=0 & x_3=L \rightarrow \\
T=0 & x_3=L \rightarrow
\end{array}
\underbrace{\begin{bmatrix} 0 & -k_n^2 & 0 & k_n^2 \\ & \text{etc} & & \\ & & & \\ & & & \end{bmatrix}}_{H}
\begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Slap into MATLAB set $\det[H] = 0$

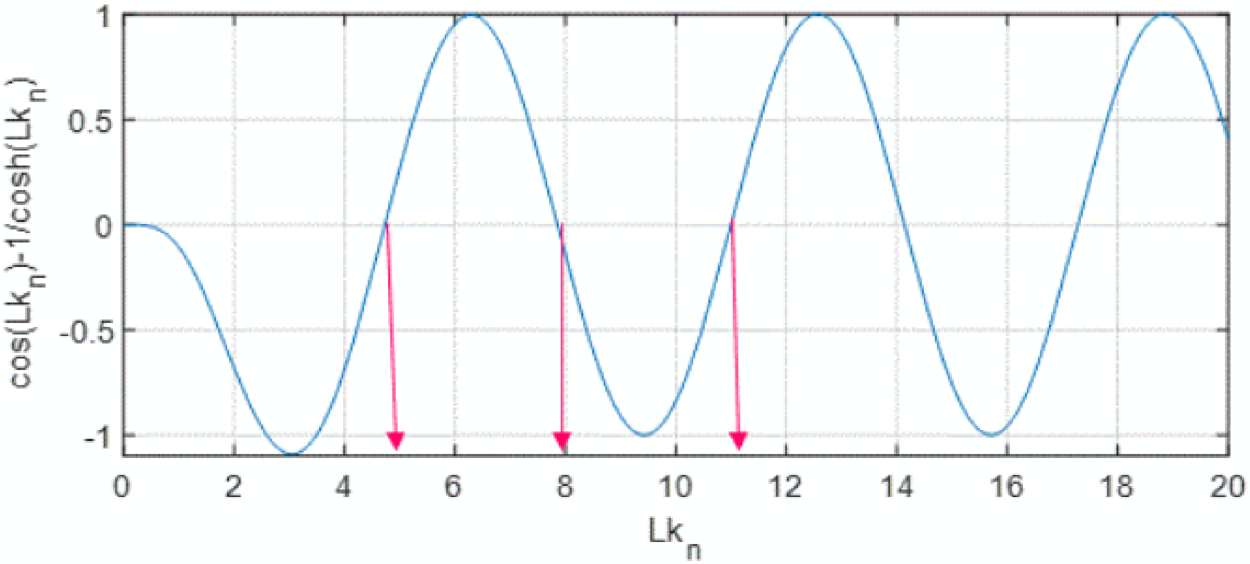
```
syms x kn L real
u = [sin(kn*x),cos(kn*x),sinh(kn*x),cosh(kn*x)]
H = [subs(diff(u,x,2),x,0);...
     subs(-diff(u,x,3),x,0);...
     subs(diff(u,x,2),x,L);...
     subs(-diff(u,x,3),x,L)]
 simplify(det(H))
 fplot(cos(x)-1/cosh(x),[0,20])
 fsolve(@(x) cos(x)-1/cosh(x),4)
 fsolve(@(x) cos(x)-1/cosh(x),8)
 fsolve(@(x) cos(x)-1/cosh(x),11)
```

```
Equation solved.

fsolve completed because the vector of function values is near zero
as measured by the default value of the function tolerance, and
the problem appears regular as measured by the gradient.

<stopping criteria details>
ans = 4.7300
Equation solved.

fsolve completed because the vector of function values is near zero
as measured by the default value of the function tolerance, and
the problem appears regular as measured by the gradient.

<stopping criteria details>
ans = 7.8532
Equation solved.

fsolve completed because the vector of function values is near zero
as measured by the default value of the function tolerance, and
the problem appears regular as measured by the gradient.

<stopping criteria details>
ans = 10.9956
```

H =

$$\begin{pmatrix} 0 & -\mathrm{kn}^2 & 0 & \mathrm{kn}^2 \\ \mathrm{kn}^3 & 0 & -\mathrm{kn}^3 & 0 \\ -\mathrm{kn}^2 \sin(L\,\mathrm{kn}) & -\mathrm{kn}^2 \cos(L\,\mathrm{kn}) & \mathrm{kn}^2 \sinh(L\,\mathrm{kn}) & \mathrm{kn}^2 \cosh(L\,\mathrm{kn}) \\ \mathrm{kn}^3 \cos(L\,\mathrm{kn}) & -\mathrm{kn}^3 \sin(L\,\mathrm{kn}) & -\mathrm{kn}^3 \cosh(L\,\mathrm{kn}) & -\mathrm{kn}^3 \sinh(L\,\mathrm{kn}) \end{pmatrix}$$

ans = $-\mathrm{kn}^{10}\,(2\cos(L\,\mathrm{kn})\cosh(L\,\mathrm{kn}) - 2)$



Equation for k_n

$$\cos k_n L \cosh k_n L - 1 = 0$$ ← Solve for k_n

$$\Rightarrow \cos k_n L - \frac{1}{\cosh k_n L} = 0$$

Solve with MATLAB $L k_1 = 4.73$

$L k_2 = 7.85$

$L k_3 = 10.995$

etc

For large k_n $\cos k_n L \approx 0 \Rightarrow k_n L = (2n+1)\frac{\pi}{2}$ $n \geq 4$

Finally $$\omega_1 = \left(\frac{4.73}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = \left(\frac{7.85}{L}\right)^2 \sqrt{\frac{EI}{\rho A}}$$

etc.