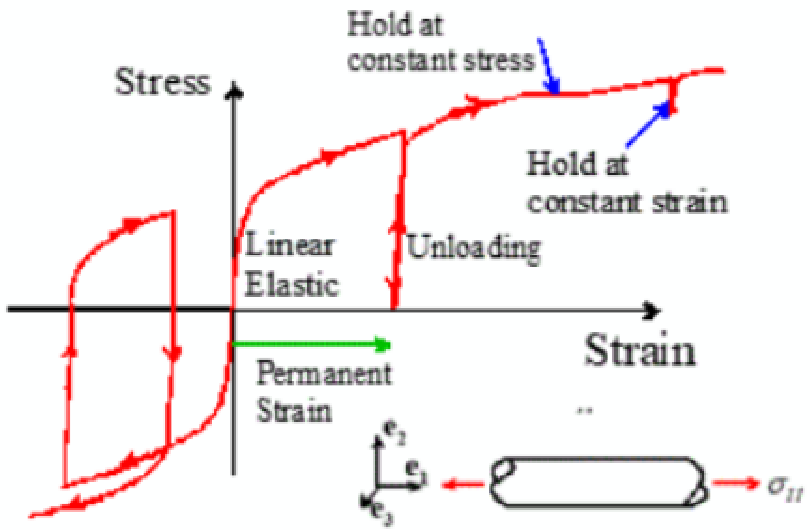
13 Plasticity

Goal: Predict behavior of metals loaded beyond yield
[similar ideas used for polymers, soils]

Applications: metal forming; crash & impact
structural analysis

13.1 Features of inelastic behavior of metals

Uniaxial Behavior

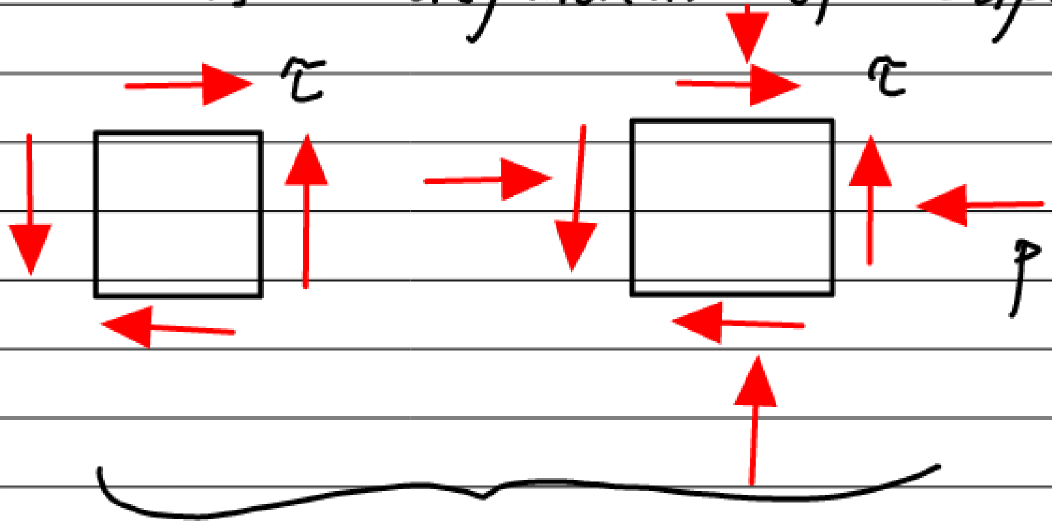


Focus on strain rates
$10^{-3} < \dot{\varepsilon} < 10^{2}$
Room temp
Strains < 20%

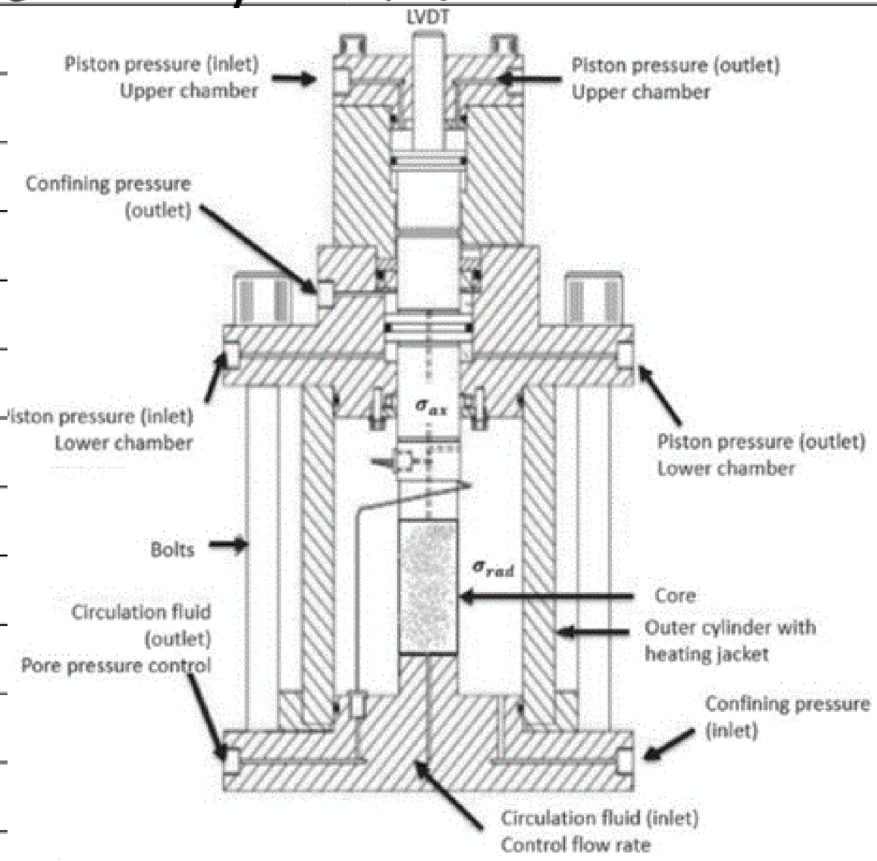
Multiaxial loading

(1) Plastic strains preserve volume $\varepsilon^p_{kk} = 0$

(2) Shear stress required to cause yield is independent of superimposed pressure



τ at yield independent of p



Levy- Mises flow law

- specifies components of plastic strain rate under multiaxial loading

Let $\{\dot{\varepsilon}_1^p, \dot{\varepsilon}_2^p, \dot{\varepsilon}_3^p\}$ be principal plastic strain rates

$\{\sigma_1, \sigma_2, \sigma_3\}$ are principal stresses

Then usually

$$\frac{\dot{\varepsilon}_1^p - \dot{\varepsilon}_2^p}{\sigma_1 - \sigma_2} = \frac{\dot{\varepsilon}_2^p - \dot{\varepsilon}_3^p}{\sigma_2 - \sigma_3} = \frac{\dot{\varepsilon}_1^p - \dot{\varepsilon}_3^p}{\sigma_1 - \sigma_3}$$

These are experimental observations

Can show that this behavior occurs because plastic flow is caused by dislocation motion obeying Schmid's law

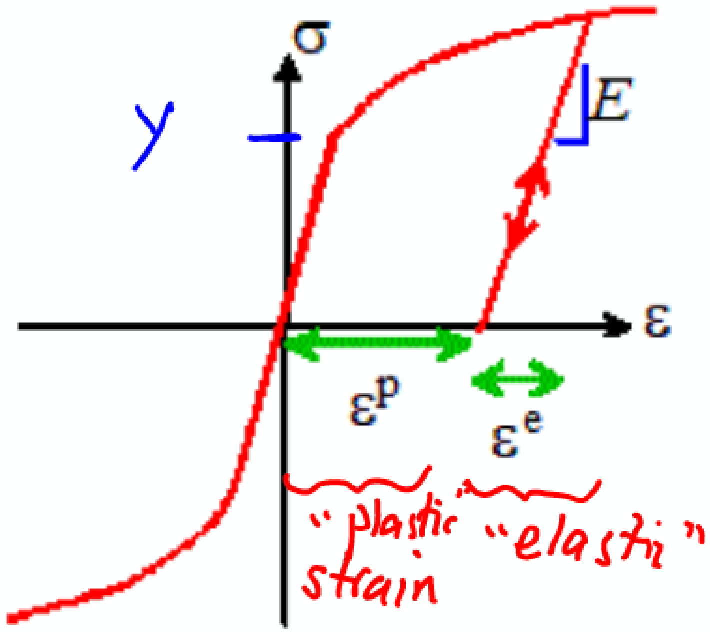
Some terminology

Yield stress: Critical stress at onset of nonlinear behavior

Elastic strain: reversible part of strain

Plastic strain: irreversible part

Strain hardening: increase of yield stress or flow stress with plastic strain

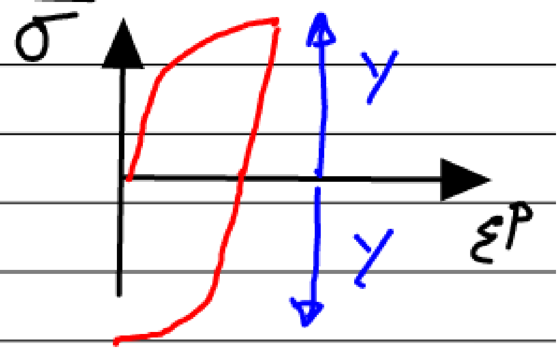


13·1 Stress-strain relations for plasticity

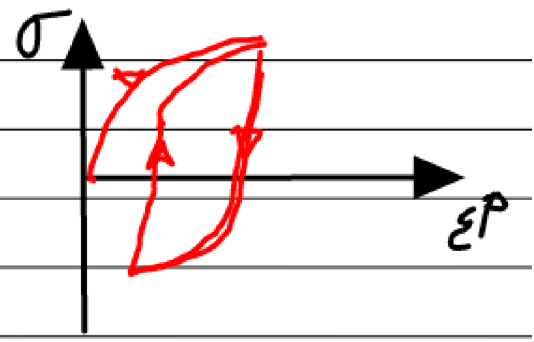
Many choices; pick the best for your application

(A) "Rate Independent" plasticity

Flow stress is independent of strain rate
At constant stress no plastic strain rate



(1) Isotropic hardening material
Hardening in tension & compression equal



(2) Kinematic hardening: tensile strain reduces compressive yield stress

- for cyclic loading

Rate independent isotropic hardening is ABAQUS' default

(B) "Rate dependent" plasticity (viscoplasticity)

Flow stress depends on strain rate.
At fixed stress nonzero plastic strain rate

- Good for creep
- Good for high rate deformation (crash, machining)

Stress-strain relations for isotropic rate independent plasticity

Complex: often need to use computer

Main ideas: (1) Yield Criterion
(2) Strain partitioning (elastic & plastic)
(3) Hardening Law
(4) Flow law

Yield Criterion

Define σ_{ij} stress

$\sigma_h = \sigma_{kk}/3 = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$ hydrostatic stress

$S_{ij} = \sigma_{ij} - \sigma_h \delta_{ij}$ deviatoric stress

$\{\sigma_1, \sigma_2, \sigma_3\}$ Principal stresses

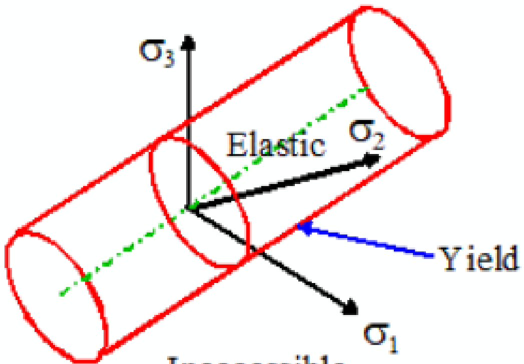
* Von Mises yield criterion

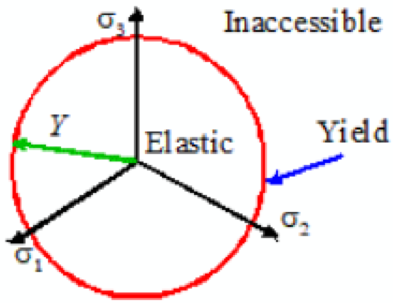
Von Mises "effective" stress $\sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}}$

or $\sigma_e = \sqrt{\left\{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]\right\}}$

Yield condition $\sigma_e < Y$ Elastic

$\sigma_e = Y$ plastic





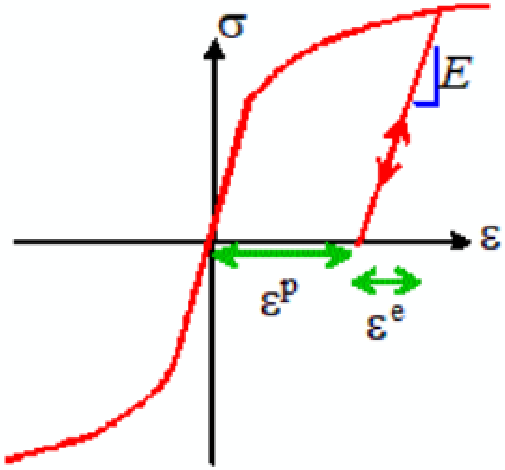
Von-Mises yield surface

[Also "Tresca" criterion]

$$\max\{(\sigma_1-\sigma_2);\ (\sigma_1-\sigma_3);\ (\sigma_2-\sigma_3)\} \begin{array}{ll} < Y & \text{elastic} \\ = Y & \text{plastic} \end{array}$$

(Good for some hand calculations)

Strain Partitioning



$$\frac{d\varepsilon_{ij}}{dt} = \underbrace{\frac{d\varepsilon_{ij}^{p}}{dt}}_{\text{Plastic}} + \underbrace{\frac{d\varepsilon_{ij}^{e}}{dt}}_{\text{Elastic}} + \underbrace{\frac{d\varepsilon_{ij}^{T}}{dt}}_{\text{Thermal}}$$

We use usual elastic stress-strain equations for $\frac{d\varepsilon_{ij}^{e}}{dt}$ $\frac{d\varepsilon_{ij}^{T}}{dt}$

$$\frac{d\varepsilon_{ij}^{e}}{dt} = \frac{1+\nu}{E}\frac{d\sigma_{ij}}{dt} - \frac{\nu}{E}\frac{d\sigma_{kk}}{dt}\delta_{ij}$$

$$\frac{d\varepsilon_{ij}^{T}}{dt} = \alpha \frac{d}{dt}(\Delta T)\,\delta_{ij}$$

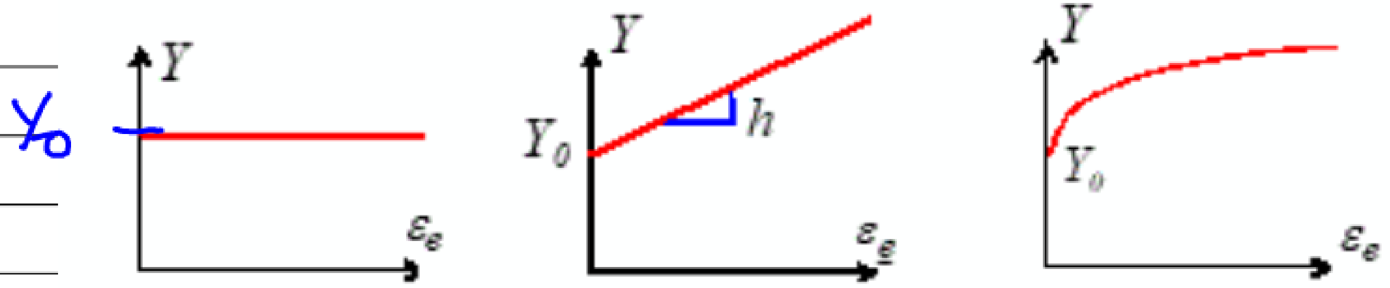
Strain Hardening Law

Predicts how yield stress increases with plastic strain

Define "Von Mises effective plastic strain"

$$\varepsilon_e = \int_0^t \sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^{p}}{dt}\frac{d\varepsilon_{ij}^{p}}{dt}}\,dt \qquad \text{(quantifies magnitude)}$$

Then make Y increase with ε_e



(a) (b) (c)

(a) "Perfect" plasticity $\quad Y = Y_0 \quad$ (const)

(b) Linear hardening $\quad Y = Y_0 + h\,\varepsilon_e$

(c) Power-law $\quad Y = Y_0 \left(1 + \dfrac{\varepsilon_e}{\varepsilon_0}\right)^{1/n}$

ε_0, h, n, Y_0 properties

Flow Rule (predicts all strain components)

$$\frac{d\varepsilon_{ij}^{p}}{dt} = \begin{cases} 0 & \sigma_e < Y(\varepsilon_e) \\ \frac{d\varepsilon_e}{dt}\left(\frac{3}{2}\frac{S_{ij}}{\sigma_e}\right) & \sigma_e = Y(\varepsilon_e) \end{cases}$$

$\frac{d\varepsilon_e}{dt}$ determined from yield condition

$$\sigma_e = Y(\varepsilon_e)$$

$$\Rightarrow \frac{d\sigma_e}{d\sigma_{ij}}\frac{d\sigma_{ij}}{dt} = \frac{dY}{d\varepsilon_e}\frac{d\varepsilon_e}{dt} \quad \text{with } \frac{d\varepsilon_e}{dt} \geq 0$$

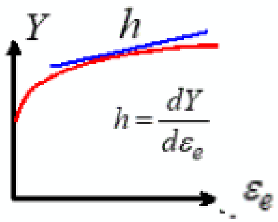
Note $\frac{d\sigma_e}{d\sigma_{ij}} = \frac{3}{2}\frac{S_{ij}}{\sigma_e}$ (if you can derive this you own index notation!)

Hence $$\frac{d\varepsilon_e}{dt} = \frac{1}{\underbrace{\frac{dY}{d\varepsilon_e}}_{\frac{dY}{d\varepsilon_e} = h}} \left\langle \frac{3}{2} \frac{S_{ij}}{\sigma_e} \frac{d\sigma_{ij}}{dt} \right\rangle$$

Where $$\langle x \rangle = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$$

Summary of elastic-plastic stress-strain relations

$$\frac{d\varepsilon_{ij}}{dt}=\begin{cases} \frac{1+\nu}{E}\left(\frac{d\sigma_{ij}}{dt}-\frac{\nu}{1+\nu}\frac{d\sigma_{kk}}{dt}\delta_{ij}\right)+\alpha\frac{d\Delta T}{dt}\delta_{ij} & \sigma_e-Y(\varepsilon_e)<0 \quad \text{(Elastic)}\\ \underbrace{\frac{1+\nu}{E}\left(\frac{d\sigma_{ij}}{dt}-\frac{\nu}{1+\nu}\frac{d\sigma_{kk}}{dt}\delta_{ij}\right)}_{d\varepsilon_{ij}^{e}/dt}+\alpha\frac{d\Delta T}{dt}\delta_{ij}+\underbrace{\frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle\frac{3}{2}\frac{S_{ij}}{\sigma_e}}_{d\varepsilon_{ij}^{p}/dt} & \sigma_e-Y(\varepsilon_e)=0 \quad \text{(Plastic)}\end{cases}$$



$$\sigma_e=\sqrt{\frac{3}{2}S_{ij}S_{ij}}=\sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]} \qquad S_{ij}=\sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\frac{d\varepsilon_e}{dt}=\sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^{p}}{dt}\frac{d\varepsilon_{ij}^{p}}{dt}}=\frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle \qquad \langle x\rangle=\begin{cases}x & x>0\\ 0 & x<0\end{cases}$$

Matrix Form

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix}=\begin{cases}\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix} & \sigma_e-Y(\varepsilon_e)<0 \quad \text{(Elastic)}\\ \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}+\alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix}+\frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle\frac{3}{2}\begin{bmatrix}(\sigma_{11}-\sigma_h)/\sigma_e\\ (\sigma_{22}-\sigma_h)/\sigma_e\\ (\sigma_{33}-\sigma_h)/\sigma_e\\ 2\sigma_{23}/\sigma_e\\ 2\sigma_{13}/\sigma_e\\ 2\sigma_{12}/\sigma_e\end{bmatrix} & \sigma_e-Y(\varepsilon_e)=0 \quad \text{(Plastic)}\end{cases}$$

$$\sigma_h=(\sigma_{11}+\sigma_{22}+\sigma_{33})/3 \qquad S_{kl}\frac{d\sigma_{kl}}{dt}=[S_{11},S_{22},S_{33},2S_{12},2S_{13},2S_{23}]\cdot\frac{d}{dt}[\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{13},\sigma_{23}]$$