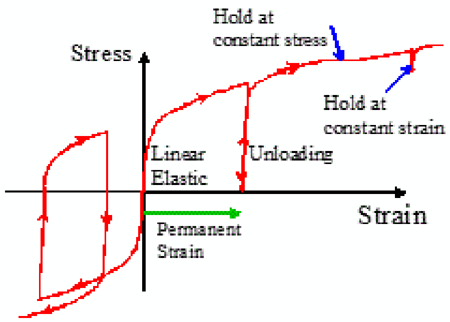
Review

Behavior of metals/polymers loaded beyond yield

Different plasticity models exist for:

- Room T quasi-static loading (rate independent/isotropic)
- High temperatures (creep) (rate dependent viscoplasticity)
- Dynamic loading (rate dependent viscoplasticity)
- Cyclic loading (kinematic hardening)
- Special models exist for soils/single xtals/polymers

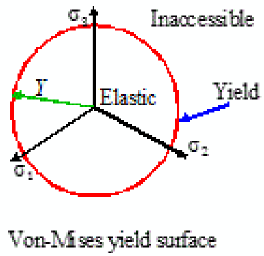


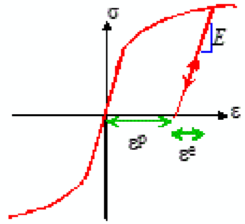
Rate Independent Plasticity Model

Yield Criterion (Von Mises) $\sigma_e < Y$ for elastic behavior

$$\sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}} = \sqrt{\frac{1}{2}\left\{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right\}}$$

$$S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$$



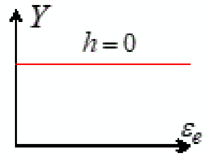


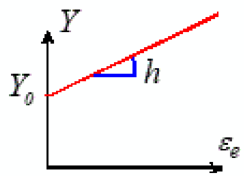
Strain Partitioning

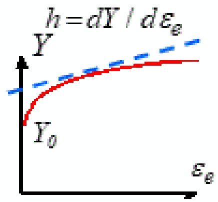
$$\frac{d\varepsilon_{ij}}{dt} = \frac{d\varepsilon_{ij}^e}{dt} + \frac{d\varepsilon_{ij}^T}{dt} + \frac{d\varepsilon_{ij}^p}{dt}$$

Hardening Rule $Y(\varepsilon_e)$

$$\frac{d\varepsilon_e}{dt} = \sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^p}{dt}\frac{d\varepsilon_{ij}^p}{dt}} = \frac{3}{2h\sigma_e}\left\langle S_{kl}\frac{d\sigma_{kl}}{dt}\right\rangle$$





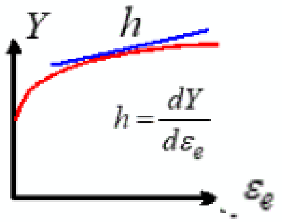


Flow Rule

$$\frac{d\varepsilon_{ij}^p}{dt} = \frac{d\varepsilon_e}{dt}\frac{3}{2}\frac{S_{ij}}{\sigma_e} \qquad \langle x\rangle = \begin{cases} x & x>0 \\ 0 & x<0 \end{cases}$$

Summary of elastic-plastic stress-strain relations

$$\frac{d\varepsilon_{ij}}{dt} = \begin{cases} \frac{1+\nu}{E}\left(\frac{d\sigma_{ij}}{dt} - \frac{\nu}{1+\nu}\frac{d\sigma_{kk}}{dt}\delta_{ij}\right) + \alpha\frac{d\Delta T}{dt}\,\delta_{ij} & \sigma_e - Y(\varepsilon_e) < 0 \quad \text{(Elastic)} \\ \underbrace{\frac{1+\nu}{E}\left(\frac{d\sigma_{ij}}{dt} - \frac{\nu}{1+\nu}\frac{d\sigma_{kk}}{dt}\delta_{ij}\right)}_{d\varepsilon_{ij}^e/dt} + \alpha\frac{d\Delta T}{dt}\,\delta_{ij} + \underbrace{\frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle\frac{3}{2}\frac{S_{ij}}{\sigma_e}}_{d\varepsilon_{ij}^p/dt} & \sigma_e - Y(\varepsilon_e) = 0 \quad \text{(Plastic)} \end{cases}$$



$$\sigma_e = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]} \qquad S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

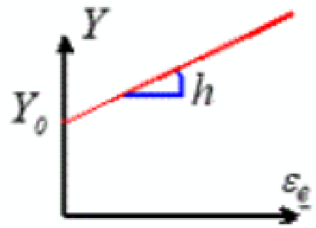
$$\frac{d\varepsilon_e}{dt} = \sqrt{\frac{2}{3}\frac{d\varepsilon_{ij}^p}{dt}\frac{d\varepsilon_{ij}^p}{dt}} = \frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle \qquad \langle x\rangle = \begin{cases} x & x>0 \\ 0 & x<0 \end{cases}$$

Matrix Form

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{13}\\ 2\varepsilon_{12}\end{bmatrix} = \begin{cases} \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} + \alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix} & \sigma_e - Y(\varepsilon_e) < 0 \quad \text{(Elastic)} \\ \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu)\end{bmatrix}\frac{d}{dt}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} + \alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\\1\\0\\0\\0\end{bmatrix} + \frac{1}{h}\left\langle\frac{3}{2}\frac{S_{kl}}{\sigma_e}\frac{d\sigma_{kl}}{dt}\right\rangle\frac{3}{2}\begin{bmatrix}(\sigma_{11}-\sigma_h)/\sigma_e\\ (\sigma_{22}-\sigma_h)/\sigma_e\\ (\sigma_{33}-\sigma_h)/\sigma_e\\ 2\sigma_{23}/\sigma_e\\ 2\sigma_{13}/\sigma_e\\ 2\sigma_{12}/\sigma_e\end{bmatrix} & \sigma_e - Y(\varepsilon_e) = 0 \quad \text{(Plastic)} \end{cases}$$

$$\sigma_h = (\sigma_{11}+\sigma_{22}+\sigma_{33})/3 \qquad S_{kl}\frac{d\sigma_{kl}}{dt} = [S_{11}, S_{22}, S_{33}, 2S_{12}, 2S_{13}, 2S_{23}]\cdot\frac{d}{dt}[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}]$$

Example: An elastic-plastic material with a linear hardening relation $Y = Y_0 + h\varepsilon_e$ is loaded in uniaxial tension with $\sigma_{11} = \sigma, \quad d\sigma / dt > 0$
Find the stress-strain relation



Stress state $\underline{\sigma} = [\sigma, 0, 0, 0, 0, 0]$

Stress measures : $\sigma_h = \sigma_{kk}/3 = \sigma/3$

Deviatoric stress $\underline{S} = [S_{11}, S_{22}, \text{etc}] = \underline{\sigma} - \sigma_h [1,1,1,0,0,0]$

$$= \left[\tfrac{2}{3}\sigma, -\tfrac{\sigma}{3}, -\tfrac{1}{3}\sigma, 0, 0, 0\right]$$

Von-Mises stress : Note $\{\sigma, 0, 0\}$ are principal stresses

$$\sigma_e = \sqrt{\tfrac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2\right]} = |\sigma|$$

Effective plastic strain rate

$$\frac{d\varepsilon_e}{dt} = \begin{cases} 0 & |\sigma| < Y_0 \\ \frac{1}{h}\left\langle \frac{3}{2}\frac{S_{ij}}{\sigma_e}\frac{d\sigma_{ij}}{dt} \right\rangle & |\sigma| > Y_0 \end{cases}$$

$$= \frac{1}{h}\frac{3}{2}\left\langle \frac{1}{|\sigma|}\left[\tfrac{2}{3}\sigma, -\frac{\sigma}{3}, -\frac{\sigma}{3}, 0, 0, 0\right]\cdot\left[\frac{d\sigma}{dt}, 0, 0, 0, 0, 0\right]\right\rangle$$

$$= \frac{1}{h}\left\langle \frac{\sigma}{|\sigma|}\frac{d\sigma}{dt} \right\rangle \qquad \langle x \rangle := \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Note $\frac{d\sigma}{dt} > 0$ here $\Rightarrow \sigma > 0$

$$= \frac{1}{h}\frac{d\sigma}{dt}$$

Strain Rate

(1) Note $\frac{d}{dt}[\varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}] = 0$

We know $\varepsilon_{22} = \varepsilon_{33}$

$$\frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\end{bmatrix} = \begin{cases} \frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{bmatrix}d\sigma/dt\\ 0\end{bmatrix} & |\sigma| < Y_0 \\ \frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{bmatrix}d\sigma/dt\\ 0\end{bmatrix} + \frac{1}{h}\frac{d\sigma}{dt}\frac{3}{2}\underbrace{\begin{bmatrix}(\sigma - \sigma/3)/|\sigma|\\ -\sigma/3\,/|\sigma|\end{bmatrix}}_{\begin{bmatrix}2/3\\ -1/3\end{bmatrix}} & \end{cases}$$

Note $\frac{d\varepsilon}{dt} = \frac{d\varepsilon}{d\sigma}\frac{d\sigma}{dt}$, cancel $\frac{d\sigma}{dt}$

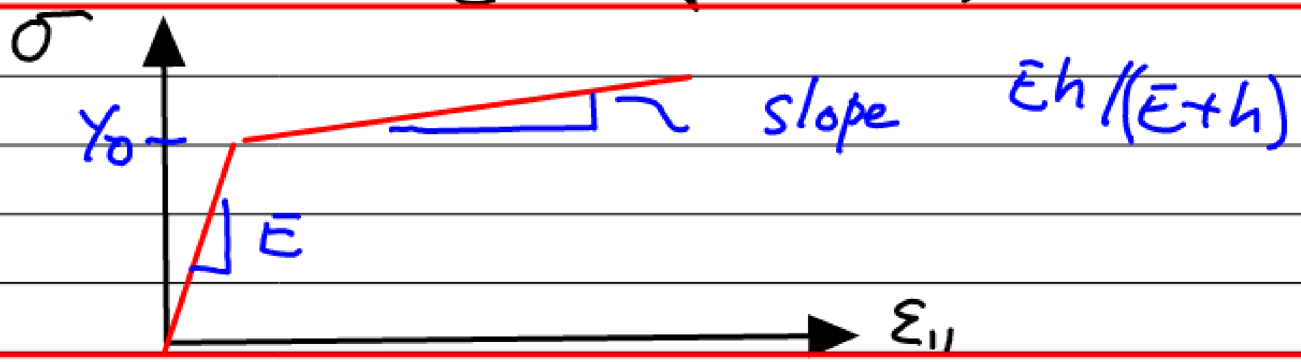
$$\frac{d\varepsilon_{11}}{d\sigma} = \begin{cases} 1/E & |\sigma| < Y_0 \\ \frac{1}{E} + \frac{1}{h} & |\sigma| > Y_0 \end{cases}$$

Now integrate $0 < \sigma < Y_0$

$Y_0 < \sigma < \infty$

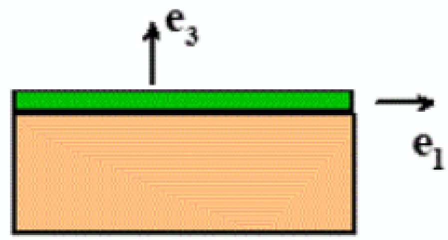
$$\varepsilon_{11} = \int_0^{Y_0} \frac{1}{E}\, d\sigma + \int_{Y_0}^{\sigma} \left(\frac{1}{E} + \frac{1}{h}\right) d\sigma$$

$$\boxed{\varepsilon_{11} = \begin{cases} \dfrac{\sigma}{E} & |\sigma| < Y_0 \quad \text{elastic} \\[2ex] \dfrac{Y_0}{E} + \left(\dfrac{1}{E} + \dfrac{1}{h}\right)(\sigma - Y_0) & |\sigma| > Y_0 \quad \text{plastic} \end{cases}}$$



$$\boxed{\text{Also} \quad \varepsilon_{22} = \begin{cases} -\nu\,\sigma/E & \sigma < Y_0 \\[2ex] -\nu\dfrac{Y_0}{E} + \left(-\dfrac{\nu}{E} - \dfrac{1}{2h}\right)(\sigma - Y_0) & \sigma > Y_0 \end{cases}}$$

Example: A thin film with Young's modulus E Poisson's ratio ν, yield stress Y_0 and thermal expansion α is bonded to a large rigid substrate (with $\alpha = 0$). It is heated by temperature change ΔT Find the value ΔT_y that will cause yield.



Elastic film on substrate was solved in L #11

Recall $\varepsilon_{11} = \varepsilon_{22} = 0$ (substrate rigid, prevents film from stretching)

Stress in film is $\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \end{bmatrix} = \frac{-E}{1-\nu} \alpha \Delta T \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

all other $\sigma_{ij} = 0$

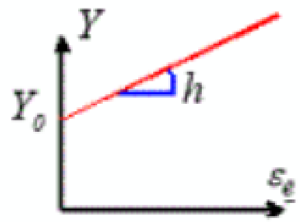
Note σ_{11}, σ_{22} are principal stresses

$\Rightarrow$ Von Mises $\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right]} = |\sigma|$

Yield criterion $\sigma_e = Y_0 \Rightarrow \frac{E\alpha\Delta T_y}{(1-\nu)} = Y_0$

$$\Rightarrow \boxed{\Delta T_y = \frac{Y_0(1-\nu)}{E\alpha}}$$

Example: Suppose that the film in the preceding problem has a linear hardening law. It is heated to a temperature $\Delta T = \beta \Delta T_y$. $\beta > 1$ Find the stress in the film.



As before $\varepsilon_{11} = \varepsilon_{22} = 0$; we expect biaxial stress
$\sigma_{11} = \sigma_{22} = \sigma$ all other $\sigma_{ij} = 0$

* Find formulas relating σ to ΔT from elastic-plastic stress-strain-temp formulas

* Find stress measures $\sigma_h = \sigma_{kk}/3 = 2\sigma/3$

Deviatoric : $\underline{S} = \left[\frac{\sigma}{3}, \frac{\sigma}{3}, -\frac{2\sigma}{3}, 0, 0, 0\right]$

Von-Mises $\sigma_e = |\sigma|$ as before

* Effective plastic strain rate :

$$\frac{d\varepsilon_e}{dt} = \begin{cases} 0 & |\sigma| < Y_0 \\ \frac{1}{h} \underbrace{\left\langle \frac{3}{2} \frac{S_{ij}}{\sigma_e} \frac{d\sigma_{ij}}{dt} \right\rangle}_{} & |\sigma| > Y_0 \end{cases}$$

$$= \frac{3}{2|\sigma|} \left\langle \left[\frac{\sigma}{3}, -\frac{\sigma}{3}, -\frac{2\sigma}{3} \right] \cdot \left[\frac{d\sigma}{dt} \quad \frac{d\sigma}{dt} \quad 0 \right] \right\rangle$$

$$= \left\langle \frac{\sigma}{|\sigma|} \frac{d\sigma}{dt} \right\rangle$$

* Now find strain rates

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\end{bmatrix}}_{=0} = \begin{cases} \frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\frac{d\sigma}{dt} + \alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\end{bmatrix} & |\sigma| < Y_0 \\ \frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\frac{d\sigma}{dt} + \alpha\frac{d\Delta T}{dt}\begin{bmatrix}1\\1\end{bmatrix} + \underbrace{\frac{d\varepsilon_e}{dt}\frac{3}{2}\begin{bmatrix}\sigma/(3|\sigma|)\\ \sigma/(3|\sigma|)\end{bmatrix}}_{=\frac{d\varepsilon_e}{dt}\frac{1}{2}\frac{\sigma}{|\sigma|}} & |\sigma| > Y_0 \end{cases}$$

Hence :

$$\frac{1-\nu}{E}\frac{d\sigma}{dt} = -\alpha\frac{d\Delta T}{dt} \qquad |\sigma| < Y_0$$

$$\frac{1-\nu}{E}\frac{d\sigma}{dt} + \underbrace{\frac{1}{2h}\frac{\sigma}{|\sigma|}\left\langle\frac{\sigma}{|\sigma|}\frac{d\sigma}{dt}\right\rangle}_{=\frac{1}{2h}\frac{d\sigma}{dt}} = -\alpha\frac{d\Delta T}{dt} \qquad |\sigma| > Y_0$$

Note $\frac{d\sigma}{dt} = \frac{d\sigma}{d\Delta T}\frac{d\Delta T}{dt}$ cancel $\frac{d\Delta T}{dt}$

$$\frac{d\sigma}{d\Delta T} = \frac{-\alpha E}{1-\nu} \qquad \Delta T < \Delta T_y \qquad \text{(elastic)}$$

$$\left(\frac{1-\nu}{E} + \frac{1}{2h}\right)\frac{d\sigma}{d\Delta T} = -\alpha \qquad \Delta T > \Delta T_y \qquad \text{(plastic)}$$

Recall: $\Delta T_y = \frac{(1-\nu)\,Y_0}{E\alpha} \qquad \Delta T = \beta \Delta T_y \quad \beta > 1$

Integrate $0 < \Delta T < \Delta T_y$ and $\Delta T_y < \Delta T < \infty$

$$\sigma = \int_0^{\Delta T_y} -\frac{\alpha E}{1-\nu}\, d\Delta T + \int_{\Delta T_y}^{\beta \cdot \Delta T_y} -\left(\frac{2Eh}{2h(1-\nu)+E}\right)\alpha\, d\Delta T$$

$$= \frac{-\alpha E}{1-\nu}\Delta T_y - \left(\frac{2\alpha E h}{2h(1-\nu)+E}\right)(\beta-1)\,\Delta T_y$$

Subst for ΔT_y

$$\boxed{\sigma = -Y_0 - \frac{2Y_0(1-\nu)h}{(2h(1-\nu)+E)}(\beta-1) \qquad \beta \geq 1}$$