School of Engineering
Brown University

EN2210: Continuum Mechanics

Midterm Examination
Wed Oct 26 2016

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1 (5 points) ______________

2. (5 points) ______________

3. (5 points) ______________

4. (5 points) ______________

5. (8 points) ______________

TOTAL (28 points) ____________

1. Let $\mathbf{R}$ be a proper orthogonal tensor satisfying $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$, det($\mathbf{R}$)=1, and let **a,b** be vectors. Using index notation, show that $(\mathbf{Ra})\times(\mathbf{Rb}) = \mathbf{R}(\mathbf{a}\times\mathbf{b})$. You will need the following result for a determinant:

$$\in_{lmn} \det(\mathbf{A}) = \in_{ijk} A_{il}A_{jm}A_{kn}$$

$$\left(\mathbf{Ra}\right)\times\left(\mathbf{Rb}\right) \equiv \in_{ijk} R_{jn}a_n R_{km}b_m = \in_{qjk} \delta_{iq} R_{jn}R_{km}a_n b_m = \in_{qjk} R_{ql}R_{il}R_{jn}R_{km}a_n b_m$$
$$= R_{il} \in_{qjk} R_{ql}R_{jn}R_{km} = R_{il} \det(\mathbf{R}) \in_{lnm} a_n b_m = R_{il} \in_{lnm} a_n b_m = \mathbf{R}\left(\mathbf{a}\times\mathbf{b}\right)$$

(5 POINTS)

2. Let **a**, **b** be two unit vectors. Find expressions for the eigenvalues and eigenvectors of $\mathbf{S} = \mathbf{a}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{a}$, in terms of $\mathbf{a},\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{b}$

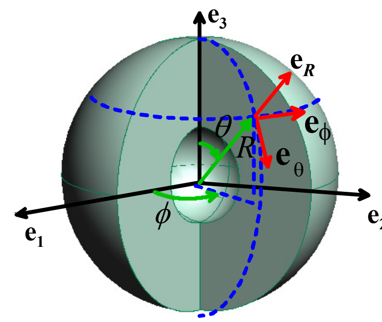
- Since $\mathbf{Su}$ must always lie in the **a**,**b** plane, one eigenvector is $\mathbf{a}\times\mathbf{b}$ and the corresponding eigenvalue is zero.
- We can construct the other eigenvectors as $\mathbf{a}+\beta\mathbf{b}$ where β is to be found. The definition of an eigenvector/value pair yields
  $\mathbf{S}(\mathbf{a}+\beta\mathbf{b}) = \left(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}\right)(\mathbf{a}+\beta\mathbf{b}) = \mathbf{a}\left[(\mathbf{a}\cdot\mathbf{b})+\beta\right]+\mathbf{b}\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right] = \gamma(\mathbf{a}+\beta\mathbf{b})$

- We can see by inspection that choosing $\beta = 1 \;\Rightarrow \mathbf{a}\left[(\mathbf{a}\cdot\mathbf{b})+1\right]+\mathbf{b}\left[1+(\mathbf{a}\cdot\mathbf{b})\right] = \left[1+(\mathbf{a}\cdot\mathbf{b})\right](\mathbf{a}+\mathbf{b})$ so $\gamma = 1+\mathbf{a}\cdot\mathbf{b}$ is an eigenvalue and $(\mathbf{a}+\mathbf{b})$ is an eigenvector. The other eigenvector must be perpendicular to this one so $\pm(\mathbf{a}-\mathbf{b})$ is the other eigenvector

  You can also work step-by-step through the algebra:
  $\left[\mathbf{a}\cdot\mathbf{b}+\beta\right]+(\mathbf{a}\cdot\mathbf{b})\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right] = \gamma\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right]$
  $(\mathbf{a}\cdot\mathbf{b})\left[\mathbf{a}\cdot\mathbf{b}+\beta\right]+\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right] = \gamma\left[\mathbf{a}\cdot\mathbf{b}+\beta\right]$
  $\Rightarrow \left[\mathbf{a}\cdot\mathbf{b}+\beta\right]^2 + (\mathbf{a}\cdot\mathbf{b})\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right]\left[\mathbf{a}\cdot\mathbf{b}+\beta\right] - (\mathbf{a}\cdot\mathbf{b})\left[\mathbf{a}\cdot\mathbf{b}+\beta\right]\left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right] - \left[1+\beta(\mathbf{a}\cdot\mathbf{b})\right]^2 = 0$
  $\Rightarrow \beta^2\left(1-(\mathbf{a}\cdot\mathbf{b})^2\right)+(\mathbf{a}\cdot\mathbf{b})^2 - 1 = 0$
  $\Rightarrow \beta = \pm 1$
  Thus
  $$\left(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}\right)(\mathbf{a}+\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}+\mathbf{b}+\mathbf{a}+\mathbf{b}(\mathbf{a}\cdot\mathbf{b}) = \left[1+(\mathbf{a}\cdot\mathbf{b})\right](\mathbf{a}+\mathbf{b})$$
  $$\left(\mathbf{a}\otimes\mathbf{b}+\mathbf{b}\otimes\mathbf{a}\right)(\mathbf{a}-\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{a}+\mathbf{b}-\mathbf{a}-\mathbf{b}(\mathbf{a}\cdot\mathbf{b}) = -\left[1-(\mathbf{a}\cdot\mathbf{b})\right](\mathbf{a}-\mathbf{b})$$
  The eigenvalues are therefore $\left[1+(\mathbf{a}\cdot\mathbf{b})\right],\; -\left[1-(\mathbf{a}\cdot\mathbf{b})\right]$

(5 POINTS)

3. The figure shows an incompressible spherical shell with inner radius a and outer radius b. The interior is being inflated at constant rate $da/dt = \alpha$.



3.1 Assume that the velocity field is radial $\mathbf{v} = v_R(r)\mathbf{e}_R$. Calculate an expression for the velocity gradient in terms of v_R , and hence show that $v_R = \alpha a^2 / R^2$.

$$\mathbf{L} = \nabla_{\mathbf{y}} \mathbf{v} = \frac{\partial v_R}{\partial R}\mathbf{e}_R \otimes \mathbf{e}_R + \frac{v_R}{R}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi\right)$$

Incompressibility requires tr(**L**)=0, which yields

$$\frac{dv_R}{dR} = -\frac{2v_R}{R} \Rightarrow \log(v_R) = -2\log(R) \Rightarrow v_R = \alpha\frac{a^2}{R^2}$$

[3 POINTS]

3.2 Suppose that a fly walks around the outer surface of the shell with constant speed w relative to the surface. Find a formula for the acceleration of the fly.

The acceleration is

$$\mathbf{a} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{y}} + \left(\nabla_{\mathbf{y}}\mathbf{v}\right)\mathbf{v} = 2\alpha^2\frac{a}{R^2}\mathbf{e}_R + \alpha\left(-\frac{a^2}{R^3}\mathbf{e}_R \otimes \mathbf{e}_R + \frac{a^2}{R^3}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi\right)\right)\left(\alpha\frac{a^2}{R^2}\mathbf{e}_R + w(\beta\mathbf{e}_\theta \pm \sqrt{1-\beta^2}\mathbf{e}_\theta\right)$$

$$= \left(2\alpha^2\frac{a}{R^2} - \alpha^2\frac{a^4}{R^5}\right)\mathbf{e}_R + 2\alpha w\frac{a^2}{R^3}(\beta\mathbf{e}_\theta \pm \sqrt{1-\beta^2}\mathbf{e}_\theta) \qquad -1 < \beta < 1$$

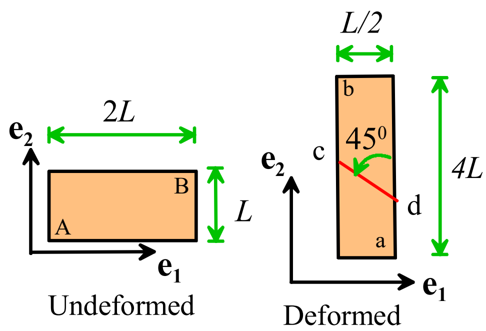
[2 POINTS]

4. The figure shows the reference and deformed configurations for a solid. The corners A and B map to *a* and *b* after deformation. Material fibers perpendicular to the plane retain their length and orientation during deformation.

The stress-strain law for the material relates the material stress $\mathbf{\Sigma}$ to the Lagrange strain $\mathbf{E}$ by

$$\mathbf{\Sigma} = 4\mathbf{E}$$

Calculate the traction acting on the plane c-d shown in the figure.



- Recall **F=VR**

$$J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = 2(\mathbf{F}^T\mathbf{F}-\mathbf{I}) \Rightarrow \boldsymbol{\sigma} = \frac{1}{J}\left(\mathbf{F}\mathbf{F}^T\mathbf{F}\mathbf{F}^T - \mathbf{F}\mathbf{F}^T\right) = \frac{2}{J}\left(\left(\mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V}\right)^2 - \mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V}\right) = \frac{2}{J}\left(\mathbf{V}^4 - \mathbf{V}^2\right)$$

$$\boldsymbol{\sigma} = 2\left(\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}^4 - \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}^2\right) = \begin{bmatrix} -3/8 & 0 \\ 0 & 24 \end{bmatrix}$$

- The traction follows as

$$\mathbf{n}\cdot\boldsymbol{\sigma} = \frac{1}{\sqrt{2}}\begin{bmatrix} -3/8 \\ 24 \end{bmatrix}$$

- Alternatively, the deformation gradient can be constructed as a sequence of stretch and rotation

$$\mathbf{F} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}$$

- The Lagrange strain follows as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left(\begin{bmatrix} 0 & 2 \\ -1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 3 & 0 \\ 0 & -3/4 \end{bmatrix}$$

  (Can also get this as $(\mathbf{U}^2 - \mathbf{I})/2$

- The material stress follows as

$$\mathbf{\Sigma} = 2\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) = \begin{bmatrix} 6 & 0 \\ 0 & -3/2 \end{bmatrix}$$

- The Cauchy stress is then

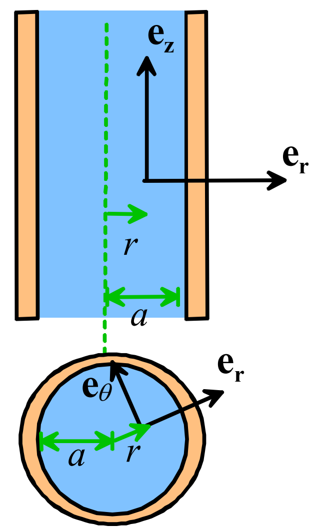
$$\boldsymbol{\sigma} = J\mathbf{F}\mathbf{\Sigma}\mathbf{F}^T = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & -3/2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 12 \\ 3/4 & 0 \end{bmatrix} = \begin{bmatrix} -3/8 & 0 \\ 0 & 24 \end{bmatrix}$$

[5 POINTS]

5. The figure shows an infinitely long cylindrical tube with a vertical axis. A fluid with mass density ρ and viscosity μ flows down the tube with steady velocity $\mathbf{v} = v(r)\mathbf{e}_z$ under the action of an axial gravitational force $\mathbf{b} = -\rho g \mathbf{e}_z$. The Cauchy stress in the fluid is

$$\boldsymbol{\sigma} = 2\mu\mathbf{D}$$

where **D** is the stretch rate.



5.1 Find an expression for the velocity gradient for the flow using the cylindrical-polar coordinate system shown in the figure.

$$\mathbf{L} = \nabla\mathbf{v} = v(r)\mathbf{e}_z \otimes \left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta + \frac{\partial}{\partial z}\mathbf{e}_z \right) = \frac{\partial v}{\partial r}\mathbf{e}_z \otimes \mathbf{e}_r$$

[2 POINTS]

5.2 Using the principle of virtual work, show that the velocity distribution must satisfy

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r} \right) = \frac{\rho g}{\mu}$$

[4 POINTS]

$$\delta P = \int_V 2\mu\mathbf{D} : \delta\mathbf{D} - \int_V \mathbf{b}\cdot\delta\mathbf{v}dV = \int_0^a \mu\frac{\partial v}{\partial r}\frac{\partial\delta v}{\partial r}2\pi r dr + \int_0^a \rho g\delta v 2\pi r dr = 0$$

Integrate by parts:

$$\int_0^a \mu\frac{\partial v}{\partial r}\frac{\partial\delta v}{\partial r}2\pi r dr = \int_0^a \mu\left[\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}r\delta v \right) - \delta v\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}r \right) \right]2\pi dr$$

$$= 2\pi\mu\left(\frac{\partial v}{\partial r}b\delta v(b) - \frac{\partial v}{\partial r}a\delta v(a) \right) - \int_0^a \mu\delta v\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} \right)2\pi r dr$$

Substitute back into PVW

$$2\pi\mu\left(\frac{\partial v}{\partial r}b\delta v(b) - \frac{\partial v}{\partial r}a\delta v(a) \right) - \int_a^b \mu\delta v\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{\rho g}{\mu} \right)2\pi r dr = 0$$

The integrand must vanish which yields the answer.

5.3. Calculate *v(r)*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)=\frac{\rho g}{\mu}\Rightarrow r\frac{\partial v}{\partial r}=\frac{1}{2}\frac{\rho g}{\mu}r^2+C\Rightarrow\frac{\partial v}{\partial r}=\frac{1}{2}\frac{\rho g}{\mu}r+\frac{C}{r}$$

$$\Rightarrow v=\frac{1}{4}\frac{\rho g}{\mu}r^2+C\log(r)+D$$

The velocity must be bounded at $r=0$ so $C=0$, and $v=0$ at $r=a$ whence $v=\frac{\rho g}{4\mu}\left(r^2-a^2\right)$

[2 POINTS]