School of Engineering
Brown University

EN2210: Continuum Mechanics

Midterm Examination
Wed Oct 7 2012

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

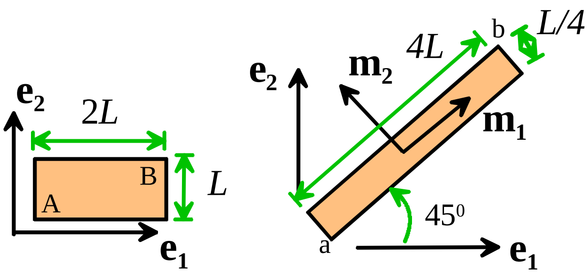
1 (14 points) _____________

2. (5 points) _____________

3. (11 points) _____________

TOTAL (30 points)___________

1.2 The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine



1.3 The right stretch tensor **U**, expressed as components in $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (A 2x2 matrix is sufficient). There is no need for lengthy calculations – you may write down the result by inspection.

The deformed configuration can be reached by a stretch parallel to the two basis vectors, followed by a rotation. These can be taken to be the two deformations in the decomposition **F=RU**

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

[2 POINTS]

1.4 The rotation tensor **R** in the polar decomposition of the deformation gradient **F=RU=VR**

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

[**2 POINTS]**

1.5 The deformation gradient, expressed as components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Try to do this without using the basis-change formulas.

The deformation gradient can be decomposed as **F=VR**, and **V** has components

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$, while **R** has the same components in both $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Therefore

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1/4 & 1/4 \end{bmatrix}$$

[3 POINTS]

1.6 If the material has a constitutive relation such that the material stress is related to Lagrange strain by:

$$\Sigma_{ij} = \mu E_{ij}$$

where μ is a material constant, show that the Cauchy stress is related to the left Cauchy-Green stretch tensor **V** by

$$\boldsymbol{\sigma} = \frac{\mu}{2J}(\mathbf{V}^4 - \mathbf{V}^2)$$

where **I** is the identity tensor.

Use the identities $\boldsymbol{\Sigma} = J\mathbf{F}^{-1}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{-T}$ $\quad \mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F}-\mathbf{I}\right) = \frac{1}{2}\left(\mathbf{U}^2-\mathbf{I}\right)$

Thus

$$
\begin{aligned}
&J(\mathbf{R}\mathbf{U})^{-1}\cdot\boldsymbol{\sigma}\cdot(\mathbf{R}\mathbf{U})^{-T} = \frac{\mu}{2}\left(\mathbf{U}^2-\mathbf{I}\right)\\
&\Rightarrow J\mathbf{U}^{-1}\mathbf{R}^T\cdot\boldsymbol{\sigma}\cdot\mathbf{R}\mathbf{U}^{-1} = \frac{\mu}{2}\left(\mathbf{U}^2-\mathbf{I}\right)\\
&\Rightarrow \boldsymbol{\sigma} = \frac{\mu}{2J}\mathbf{R}\left(\mathbf{U}^4-\mathbf{U}^2\right)\mathbf{R}^T = \frac{\mu}{2J}\left(\mathbf{I}-\mathbf{R}\mathbf{U}^2\mathbf{R}^T\right)\\
&= \frac{\mu}{2J}\left(\mathbf{V}\mathbf{R}\mathbf{U}\mathbf{U}\mathbf{R}^T\mathbf{V}-\mathbf{V}\mathbf{R}\mathbf{R}^T\mathbf{V}\right) = \frac{\mu}{2J}(\mathbf{V}^4-\mathbf{V}^2)
\end{aligned}
$$

[5 POINTS]

1.7 Hence, determine an expression for the Cauchy stress components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ and write down the values of the principal Cauchy stresses.

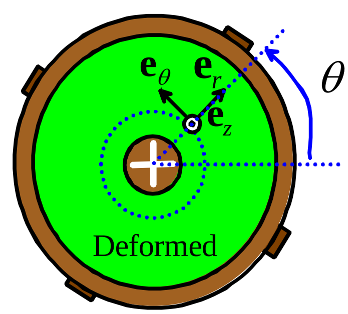
For the problem at hand $\mathbf{V} = \begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$ $\quad J = 1/2$ and so

$$
\boldsymbol{\sigma} = \mu\begin{bmatrix} 4 & 0 \\ 0 & 1/16 \end{bmatrix}\left(\begin{bmatrix} 4 & 0 \\ 0 & 1/16 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mu\begin{bmatrix} 12 & 0 \\ 0 & \dfrac{-15}{256} \end{bmatrix}
$$

[2 POINTS]

2. The figure shows the shear viscometer considered in a recent homework problem. The stress distribution in the fluid is known to have the form $\boldsymbol{\sigma} = \sigma_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$, where $\sigma_{r\theta}$ is a function of *r*. Assume that the stress state is in static equilibrium and neglect body forces. By considering a virtual velocity field of the form $\mathbf{v} = \delta v_\theta(r)\mathbf{e}_\theta$, use the principle of virtual work to show that the equilibrium equation for the stress field reduces to

$$
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0
$$



The virtual work principle is

$$
\int_V \boldsymbol{\sigma} : \delta\mathbf{D} = \int_S \mathbf{t}^* \cdot \delta\mathbf{v}
$$

The velocity gradient is

$$
\mathbf{L} = \begin{bmatrix} 0 & -\dfrac{\delta v_\theta}{r} \\ \dfrac{\partial \delta v_\theta}{\partial r} & 0 \end{bmatrix}
$$

and so

$$
\boldsymbol{\sigma} : \delta\mathbf{D} = \boldsymbol{\sigma} : \delta\mathbf{L} = \sigma_{r\theta}\left(\frac{\partial \delta v_\theta}{\partial r} - \frac{v_\theta}{r}\right)
$$

The virtual work equation therefore reduces to

$$\int_a^b \sigma_{r\theta}\left(\frac{\partial \delta v_\theta}{\partial r}-\frac{v_\theta}{r}\right)2\pi r dr = 2\pi\delta v_\theta(b)\sigma_{r\theta}(b)-2\pi a\delta v_\theta(a)\sigma_{r\theta}(a)$$

Integrate the first term by parts

$$\int_a^b r\sigma_{r\theta}\frac{\partial \delta v_\theta}{\partial r} = b\delta v_\theta(b)\sigma_{r\theta}(b)-a\delta v_\theta(a)\sigma_{r\theta}(a)-\int_a^b \frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right)\delta v_\theta dr$$

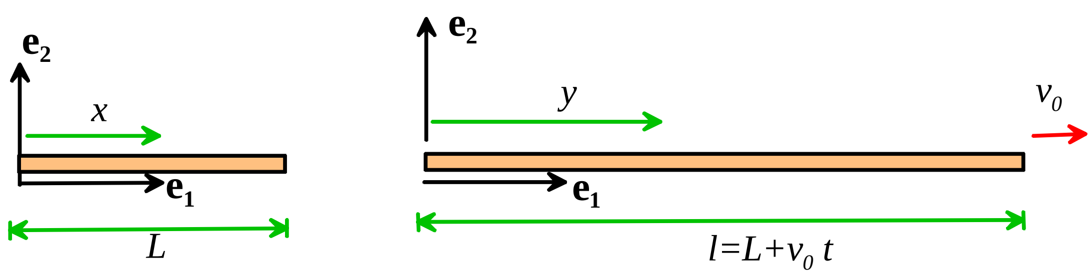
Hence

$$-\int_a^b\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right)+\frac{\sigma_{r\theta}}{r}\right)\delta v_\theta r dr = 0 \qquad \forall \delta v_\theta$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right)+\frac{\sigma_{r\theta}}{r} = \frac{\partial \sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}=0$$

[5 POINTS]

3. A rubber band has initial length *L*. One end of the band is held fixed. For time *t>0* the other end is pulled at constant speed v_0. Following the usual convention, let *x* denote position in the reference configuration, and let *y* denote position in the deformed configuration. Assume one dimensional deformation.



3.1 Write down the position *y* of a material particle as a function of its initial position *x* and time *t*.

$$y=\frac{L+v_0 t}{L}x$$

[1 POINT]

3.2 Hence, determine the velocity distribution as both a function of *x* and a function of *y*.

$$v=\frac{dl}{dt}\frac{x}{L}=\frac{v_0 x}{L}=\frac{v_0 y}{L+v_0 t}$$

[2 POINTs]

3.3 Find the deformation gradient (you only need to state the one nonzero component)

$$F=\frac{L+v_0 t}{L}$$

[1 POINT]

3.4 Find the velocity gradient

$$\frac{dv}{dy}=\frac{v_0}{L+v_0 t}$$

[1 POINT]

3.5 Suppose that a fly walks along the rubber band with speed *w* relative to the band. Calculate the acceleration of the fly as a function of time and other relevant variables.

$$a=\frac{\partial v}{\partial t}+\frac{\partial v}{\partial y}\left(v+w\right)=-\frac{v_0^2 y}{(L+v_0t)^2}+\frac{v_0}{L+v_0t}\left(\frac{v_0 y}{L+v_0t}+w\right)=\frac{v_0 w}{L+v_0t}$$

[2 POINTS]

3.6 Suppose that the fly is at *x=y=0* at time *t=0*. Find how long it takes for the fly to walk to the other end of the rubber band, in terms of *L*, v_0 and *w*. It is easiest to do this by calculating dx/dt for the fly.

$$\begin{aligned}
&y=F_{11}x\Rightarrow\frac{dy}{dt}=\dot{F}_{11}x+F_{11}\dot{x}\\
&\Rightarrow\frac{dx}{dt}=F_{11}^{-1}\left(\frac{dy}{dt}-\dot{F}_{11}x\right)=\frac{L}{L+v_0t}\left(\frac{v_0}{L}x+w-\frac{v_0}{L}x\right)=\frac{w}{1+v_0t/L}\\
&\Rightarrow x=\frac{wL}{v_0}\log(1+v_0t/L)\\
&\Rightarrow t(x=L)=\frac{L}{v_0}\left(\exp(v_0/w)-1\right)
\end{aligned}$$

[4 POINTS]

Also as a check, we see that

$$\begin{aligned}
&y=F_{11}x=\left(1+\frac{v_0t}{L}\right)\frac{wL}{v_0}\log(1+v_0t/L)\\
&\Rightarrow\frac{dy}{dt}=w\log(1+v_0t/L)+w\\
&\Rightarrow\frac{d^2y}{dt^2}=\frac{v_0w}{L+v_0t}
\end{aligned}$$