School of Engineering
Brown University

# EN2210: Continuum Mechanics

## Final Examination
## Wed Dec 19 2012

**NAME:** ________________________________________

## General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

## Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

_____________

**1 (15 points)** _____________

**2. (5 points)** _____________

**3. (15 points)** _____________

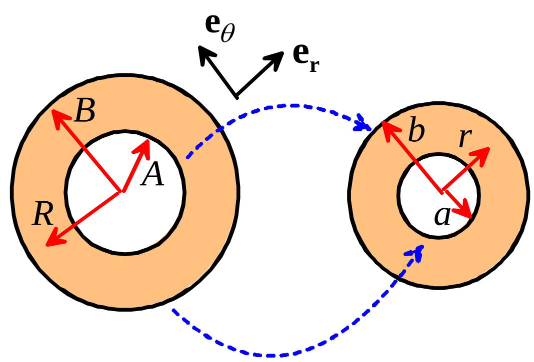
**4. (20 points)** _____________

**5. (5 points)** _____________

**TOTAL (60 points)** ____________

**1.** A rubber tube with internal radius $A$ and external radius $B$ is turned inside out, so that the surfaces at $R=A$, $R=B$ now lie at deformed radius $r=b$, $r=a$, respectively. These surfaces are free of traction. Assume that plane cross-sections of the tube remain plane, ***and that the length of the cylinder does not change***. Note that a cross section at distance $Z$ along the axis moves to a new position $z=-Z$ after deformation. The tube may be idealized as an incompressible, neo-Hookean material with stress-strain relation

$$\boldsymbol{\sigma} = \mu\mathbf{B} + p(r)\mathbf{I}$$




1.1 By considering the volumes of material in the annular regions between $r$ and $a$, and between $R$ and $B$, write down an expression for the radial position $r$ of a material particle that starts at radius $R$ in the tube before deformation.

**[2 POINTS]**

1.2 Show that the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{bmatrix} -(R/r) & 0 & 0 \\ 0 & (r/R) & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and hence write down an expression for the stress in the tube as a function of $r$, $R$ and $p$.

**[3 POINTS]**

1.3 By considering a virtual velocity $\delta \mathbf{v} = v(r)\mathbf{e}_r$ where $v(r)$ is a continuously differentiable function, show that the principle of virtual work requires that

$$\int_a^b \left[ \frac{d}{dr}\left\{ r\left( \frac{R^2}{r^2} + p \right) \right\} - \left( \frac{r^2}{R^2} + p \right) \right] v(r)\,dr = 0 \qquad \forall v(r)$$

**[5 POINTS]**

1.4 Hence show that the radii of the tube after deformation must satisfy the equations

$$\log(B^2 / a^2) + \frac{B^2}{a^2} = \log(A^2 / b^2) + \frac{A^2}{b^2}$$

$$b^2 - a^2 = B^2 - A^2$$

**[5 POINTS]**

**2.** Starting with the local form of the second law of thermodynamics and mass conservation

$$\rho \left.\frac{\partial s}{\partial t}\right|_{\mathbf{x}=const} + \frac{\partial (q_i / \theta)}{\partial y_i} - \frac{q}{\theta} \geq 0 \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

(the symbols have their usual meaning), derive the statement of the second law for a control volume

$$\frac{\partial}{\partial t}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta} dA - \int_R \frac{q}{\theta} dV \geq 0$$

**[5 POINTS]**

**3.** The deformation of a viscoelastic material is modeled by representing the deformation gradient $\mathbf{F}$ of a material element as a sequence of an irreversible deformation $\mathbf{F}^p$, followed by a reversible (elastic) deformation $\mathbf{F}^e$, so that $\mathbf{F}=\mathbf{F}^e\mathbf{F}^p$. The Helmholtz free energy $\psi(\mathbf{F}^e,\theta)$ of the material is assumed to be a function of $\mathbf{F}^e$ and temperature $\theta$ only.

3.1 Show that the velocity gradient $\mathbf{L}$ can be decomposed into elastic and plastic parts as

$$\mathbf{L}=\mathbf{L}^e+\mathbf{L}^p \qquad \mathbf{L}^e=\frac{d\mathbf{F}^e}{dt}\mathbf{F}^{e-1} \qquad \mathbf{L}^p=\mathbf{F}^e\frac{d\mathbf{F}^p}{dt}\mathbf{F}^{p-1}\mathbf{F}^{e-1}$$

**[3 POINTS]**

3.2 Show that the dissipation inequality

$$\sigma_{ij}D_{ij}-\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i}-\rho\left(\frac{\partial\psi}{\partial t}+s\frac{\partial\theta}{\partial t}\right)\geq 0$$

requires that the Cauchy stress is related to the free energy by

$$JF_{kj}^{e-1}\sigma_{ji}=\rho_0\frac{\partial\psi}{\partial F_{ik}^e}$$

(where $\rho_0$ is the mass per unit reference volume) and that the plastic part of the velocity gradient must satisfy

$$\sigma_{ij}L_{ij}^p\geq 0$$

**[3 POINTS]**

3.3 Assume that $\mathbf{F}^e$ and $\mathbf{F}^p$ transform under a change of observer according to $\mathbf{F}^{e*} = \mathbf{Q}\mathbf{F}^e \qquad \mathbf{F}^{p*} = \mathbf{F}^p$. Verify that the transformation is consistent with the transformation of deformation gradient $\mathbf{F}$ under an observer change, and determine expressions for $\mathbf{L}^{e*}, \mathbf{L}^{p*}$ in terms of $\mathbf{Q}$ and $\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$.

**[3 POINTS]**

3.4 Consider a constitutive relation in which the plastic velocity gradient is given by

$$L_{ij}^p = \eta\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

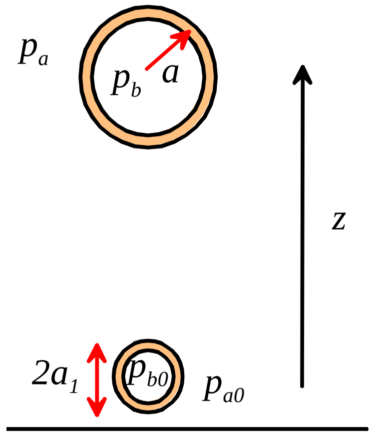
Show that if $\det(\mathbf{F}^p) = 1$ at time $t=0$, then $\det(\mathbf{F}^p) = 1$ for all $t>0$. (Hint: consider $L_{kk}^p$, and recall $L_{ij}^p = F_{ik}^e \dfrac{dF_{kl}^p}{dt} F_{lm}^{p-1} F_{mj}^{e-1}$ )

**[3 POINTS]**

3.5 Show that the constitutive relation in 3.4 satisfies both frame indifference and the dissipation inequality (assume $\eta > 0$).

**[3 POINTS]**

**4.** The goal of this problem is to derive an expression for the maximum height reached by a rubber Helium balloon that is released from the surface of the earth. Make the following assumptions:




- The balloon has total mass (the rubber, plus the He inside) $m$
- The balloon is a thin walled sphere, and prior to inflation has wall thickness $t_0$ and radius $a_0$.
- At the earth's surface the balloon has radius $a_1$ and internal pressure $p_{b0}$
- Both He and the air can be idealized as ideal gases
- The balloon can be idealized as an incompressible, neo-Hookean solid with Cauchy stress-stretch relation $\boldsymbol{\sigma} = \mu \mathbf{B} + p\mathbf{I}$
- The air temperature $\theta$ is constant (i.e does not vary with altitude), and the balloon is always in thermal equilibrium with the air.

4.1 Given that the air pressure has magnitude $p_{a0}$ at the earth's surface, calculate the variation of air pressure $p_a$ and density $\rho_a$ with height $z$ above the earth's surface, in terms of gravitational acceleration $g$, (constant) temperature $\theta$ and the gas constant for air $R_a$.

**[3 POINTS]**

4.2 Show that the radius *a* for which the balloon is neutrally buoyant is related to air pressure $p_a$ by

$$p_a = \frac{3mR_a\theta}{4\pi a^3}$$

**[2 POINTS]**

4.3 Consider equilibrium of the thin walled spherical balloon. Using the thin-walled pressure vessel approximation, find an expression for the Cauchy stress components $\sigma_{\theta\theta}$ in the balloon, in terms of the internal pressure in the balloon $p_b$, the external air pressure $p_a$, the deformed wall thickness *t* and the radius of the balloon *a*. (you can assume $\sigma_{rr} \approx 0$)

**[2 POINTS]**

4.4 By considering the lengths of infinitesimal radial and circumferential material fibers, write down the components of deformation gradient in the balloon (in spherical polar coordinates), in terms of the initial and deformed wall thicknesses $t_0, t$, and the undeformed and deformed radii of the balloon $a_0, a$. Neglect variations through the thickness of the wall.

**[2 POINTS]**

4.5 Use the incompressibility condition to calculate $t$ in terms of $t_0, a_0, a$. Use a thin walled approximation.

**[1 POINT]**

4.6 Use the constitutive equation to find an expression for the Cauchy stress $\sigma_{\theta\theta}$ in terms of $\mu, a, a_0$. Assume $\sigma_{rr} = 0$ and neglect variations through the thickness of the wall.

**[2 POINTS]**

4.7 Given that the pressure in the balloon is $p_{b0}$ at the surface of the earth, and the balloon has radius $a_1$ at the surface of the earth, use mass conservation to show that at altitude the internal pressure in the balloon is related to its radius *a* by

$$p_b = \frac{a_1^3}{a^3} p_{b0}$$

**[2 POINTS]**

4.8 Hence, show that the radius of the neutrally buoyant balloon satisfies the equation

$$\left( p_{b0} - \frac{3mR_a\theta}{4\pi a_1^3} \right) = \frac{2\mu t_0}{a_1} \left( \frac{a}{a_1} \right)^2 \left\{ 1 - \left( \frac{a_0}{a} \right)^6 \right\}$$

**[4 POINTS]**

4.9 Assuming that $(a_0 / a)^6 << 1$, find an expression for the altitude $z$ of the balloon when it is neutrally buoyant.

**[2 POINTS]**

**5.** An incompressible Newtonian viscous fluid with viscosity $\eta$ and density $\rho$ occupies the region $y_2 < 0$. At time *t=0* the fluid has velocity distribution $v_1 = u_0 \exp(-y_2^2 / b^2)$ with all other velocity components zero. The goal of this problem is to calculate the velocity in the fluid for *t=0*. Gravity and pressure variations in the fluid may be neglected.

5.1 Explain what is meant by 'Stokes flow' and state the form of the Navier-Stokes equation for conditions where the Stokes approximation holds.

**[2 POINTS]**

5.2 By considering a velocity field of the form $v_1 = u_0 f(t) \exp(-y_2^2 f(t)^2 / b^2)$, calculate the variation of velocity in the fluid with position and time for *t>0*.

**[3 POINTS]**