School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 1: Index Notation; basic tensor operations
Due 4pm Wednesday Sept 19th

Before attempting problems 1-7, read through the online notes summarizing the rules of index notation for vectors and tensors

1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

   (a) $\sigma_{ij} = C_{klij}\varepsilon_{kl}$ (b) $\in_{kkk} = 0$ (c) $\dfrac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \dfrac{\partial^2 u_i}{\partial t^2}$ (d) $\in_{ijk}\in_{ijk} = 6$

2. Match the meaning of each index notation expression shown below with an option from the list

   (a) $\lambda = T_{ij}S_{ij}$ (b) $E_{ij} = T_{ik}S_{kj}$ (c) $E_{ij} = S_{ki}T_{kj}$ (d) $a_i = \in_{kij} b_j c_k$ (e) $\lambda = a_i b_i$

   (f) δ_{ij} (g) $T_{ij}m_j = \lambda m_i$ (h) $a_i = S_{ij}b_j$ (i) $A_{ki}A_{kj} = \delta_{ij}$ (j) $A_{ij} = A_{ji}$

   (1) Product of two tensors
   (2) Product of the transpose of a tensor with another tensor
   (3) Cross product of two vectors
   (4) Product of a vector and a tensor
   (5) Components of the identity tensor
   (6) Equation for the eigenvalues and eigenvectors of a tensor
   (7) Contraction of a tensor
   (8) Dot product of two vectors
   (9) The definition of an orthogonal tensor
   (10) Definition of a symmetric tensor

3. Let $R = \sqrt{x_k x_k}$. Calculate $\dfrac{\partial R}{\partial x_i}$ and $\dfrac{\partial^2 R}{\partial x_i \partial x_j}$.

4. Let $S_{ij} = P_{ij} - P_{kk}\delta_{ij}/3$. Calculate S_{kk} (a tensor with this property is called *deviatoric*, and **S** is called the *deviatoric part* of **P**)

5. Let $R_{ij} = \cos\theta\delta_{ij} + n_i n_j(1-\cos\theta) - \sin\theta \in_{ijk} n_k$ where n_k are the components of a unit vector. Calculate $R_{ik}R_{jk}$.

6. Show (multiply out the product of permutation symbols) that

$$\det(\mathbf{S}) = \frac{1}{6}\in_{ijk}\in_{lmn} S_{il}S_{jm}S_{kn} = \frac{1}{6}S_{ii}\left(S_{jj}S_{kk} - 3S_{kj}S_{jk}\right) + \frac{1}{3}S_{ji}S_{kj}S_{ik}$$
$$= \frac{1}{6}trace(\mathbf{S})(trace(\mathbf{S})^2 - 3\mathbf{S}\cdot\cdot\mathbf{S}) + \frac{1}{3}\mathbf{S}\cdot\cdot(\mathbf{S}\cdot\mathbf{S})$$

7. Let $J = \det(\mathbf{S})$. Show that $\dfrac{\partial J}{\partial S_{mn}} = J S_{nm}^{-1}$. (this is a very useful result – we often need to differentiate the determinant of a tensor when working with constitutive equations for materials, for example)

8. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Vector **u** has components $(1,2,0)$ in this basis, while tensors **S** and **T** have components

$$\mathbf{T} \equiv \begin{bmatrix} 1 & \sqrt{6} & 0 \\ \sqrt{6} & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad \mathbf{S} \equiv \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 6 \end{bmatrix}$$

   a. Calculate the components of the following vectors and tensors

$$\mathbf{v} = \mathbf{T}\mathbf{u} \quad \mathbf{v} = \mathbf{u} \cdot \mathbf{T} \quad \mathbf{V} = \mathbf{S} + \mathbf{T} \quad \mathbf{V} = \mathbf{S} \cdot \mathbf{T} \quad \mathbf{V} = \mathbf{S}^T$$

   b. Find the eigenvalues and the components of the eigenvectors of **T**.
   c. Denote the three (unit) eigenvectors of **T** by $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ (It doesn't matter which eigenvector is which, but be sure to state your choice clearly).
   d. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a new Cartesian basis. Write down the components of **T** in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. (Don't make this hard: in the new basis, **T** must be diagonal, and the diagonal elements must be the eigenvalues. Do you see why this is the case? You just need to get them in the right order!)
   e. Calculate the components of **S** in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

9. Let **S** and **T** be tensors with components

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 2 & -2 \\ -1 & 4 & 3 \\ 1 & 2 & 6 \end{bmatrix} \qquad \mathbf{S} \equiv \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

   a. Calculate $\mathbf{S} : \mathbf{T}$ and $\mathbf{S} \cdot\cdot \mathbf{T}$
   b. Calculate trace(**S**) and trace(**T**)

10. Show that the inner product of two tensors is invariant to a change of basis.

11. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Let **R** be a proper orthogonal tensor, and let $\mathbf{m}_i = \mathbf{R}\mathbf{e}_i$
   a. Show that $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ is also a Cartesian basis (i.e. show that $\mathbf{m}_i$ are orthogonal unit vectors).
   b. Let $R_{ij}^{(\mathbf{e})}, R_{ij}^{(\mathbf{m})}$ denote the components of **R** in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, respectively. Show that $R_{ij}^{(\mathbf{e})} = R_{ij}^{(\mathbf{m})}$.