School of Engineering
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EN2210: Continuum Mechanics

Homework 1: Index Notation; basic tensor operations
Solutions

Before attempting problems 1-7, read through the online notes summarizing the rules of index notation for vectors and tensors

1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

   (a) $\sigma_{ij} = C_{klij}\varepsilon_{kl}$ (b) $\in_{kkk} = 0$ (c) $\dfrac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \dfrac{\partial^2 u_i}{\partial t^2}$ (d) $\in_{ijk}\in_{ijk} = 6$

   (a) – OK. (b) – no – the index k is repeated three times. (c) OK; (d) OK.

[3 points]

2. Match the meaning of each index notation expression shown below with an option from the list

   (a) $\lambda = T_{ij}S_{ij}$ (b) $E_{ij} = T_{ik}S_{kj}$ (c) $E_{ij} = S_{ki}T_{kj}$ (d) $a_i = \in_{kij} b_j c_k$ (e) $\lambda = a_i b_i$

   (f) δ_{ij} (g) $T_{ij}m_j = \lambda m_i$ (h) $a_i = S_{ij}b_j$ (i) $A_{ki}A_{kj} = \delta_{ij}$ (j) $A_{ij} = A_{ji}$

   (1) Product of two tensors
   (2) Product of the transpose of a tensor with another tensor
   (3) Cross product of two vectors
   (4) Product of a vector and a tensor
   (5) Components of the identity tensor
   (6) Equation for the eigenvalues and eigenvectors of a tensor
   (7) Contraction of a tensor
   (8) Dot product of two vectors
   (9) The definition of an orthogonal tensor
   (10) Definition of a symmetric tensor

(1) is (b); (2) is (c); (3) is (d); (4) is (h); (5) is (f); (6) is (g); (7) is (a); (8) is (e); (9) is (i); (10) is (j).

[5 POINTS]

3. Let $R = \sqrt{x_k x_k}$. Calculate $\dfrac{\partial R}{\partial x_i}$ and $\dfrac{\partial^2 R}{\partial x_i \partial x_j}$.

   Recall that $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ and use the chain rule.

$$\frac{\partial R}{\partial x_j} = \frac{1}{2R}\frac{\partial x_i x_i}{\partial x_j} = \frac{1}{2R}\left(\delta_{ij}x_i + x_i\delta_{ij}\right) = \frac{x_i}{R}$$

$$\frac{\partial^2 R}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{R} + x_i \frac{\partial}{\partial x_j}\left(\frac{1}{R}\right) = \frac{\delta_{ij}}{R} - \frac{x_i}{R^2}\frac{\partial R}{\partial x_j} = \frac{\delta_{ij}}{R} - \frac{x_i x_j}{R^3}$$

[2 POINTS]

4. Let $S_{ij} = P_{ij} - P_{kk}\delta_{ij}/3$. Calculate S_{kk} (a tensor with this property is called *deviatoric*, and **S** is called the *deviatoric part* of **P**)

   Recall that $\delta_{kk} = 3$ so that $S_{kk} = P_{kk} - P_{kk}\delta_{mm}/3 = 0$

[1 POINT]

5. Let $R_{ij} = \cos\theta\delta_{ij} + n_i n_j(1-\cos\theta) - \sin\theta \in_{ijk} n_k$ where n_k are the components of a unit vector. Calculate $R_{ik}R_{jk}$. (You should know what the answer is, of course, but see if you can verify the result using index notation manipulations).

$$
\begin{aligned}
R_{ik}R_{jk} &= \left(\cos\theta\delta_{ik} + n_i n_k(1-\cos\theta) - \sin\theta \in_{ikp} n_p\right)\left(\cos\theta\delta_{jk} + n_j n_k(1-\cos\theta) - \sin\theta \in_{jkq} n_q\right) \\
&= \cos^2\theta\delta_{ik}\delta_{kj} + n_i n_k n_j n_k(1-\cos\theta)^2 + \sin^2\theta \in_{ikp} n_p \in_{jkq} n_q \\
&\quad + \cos\theta(1-\cos\theta)(\delta_{ik}n_j n_k + n_i n_k\delta_{jk}) - \cos\theta\sin\theta(\delta_{ik} \in_{jkq} n_q + \delta_{jk} \in_{ikp} n_p) \\
&\quad - (1-\cos\theta)\sin\theta(n_i n_k \in_{jkq} n_q + n_j n_k \in_{ikp} n_p)
\end{aligned}
$$

Now simplify by noting $n_k n_k = 1$ and $\delta_{ij}n_j = n_i$

$$
\begin{aligned}
R_{ik}R_{jk} &= \cos^2\theta\delta_{ij} + n_i n_j(1-\cos\theta)^2 + \sin^2\theta \in_{ikp} n_p \in_{jkq} n_q \\
&\quad + \cos\theta(1-\cos\theta)(n_j n_i + n_i n_j) - \cos\theta\sin\theta(\in_{jiq} n_q + \in_{ijp} n_p) \\
&\quad - (1-\cos\theta)\sin\theta(n_i n_k \in_{jkq} n_q + n_j n_k \in_{ikp} n_p)
\end{aligned}
$$

Note that $n_k \in_{jkq} n_q = 0$ (expand out in full, or note that this represents **n** crossed with itself) and $(\in_{jiq} n_q + \in_{ijp} n_p) = (\in_{jiq} n_q - \in_{jip} n_p) = 0$, and recall $\in_{ijk}\in_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{mk}$, so that

$$
\begin{aligned}
R_{ik}R_{jk} &= \cos^2\theta\delta_{ij} + n_i n_j(1-\cos\theta)^2 + \sin^2\theta(\delta_{ij}\delta_{pq} - \delta_{ip}\delta_{jq})n_p n_q + 2\cos\theta(1-\cos\theta)n_j n_i \\
&= \delta_{ij}(\sin^2\theta + \cos^2\theta) + n_i n_j(1 - 2\cos\theta + \cos^2\theta - \sin^2\theta + 2\cos\theta - 2\cos^2\theta) = \delta_{ij}
\end{aligned}
$$

This verifies that R_{ij} is indeed orthogonal.

[4 POINTS]

6. Show that $\det(\mathbf{S}) = \frac{1}{6}\in_{ijk}\in_{lmn} S_{il}S_{jm}S_{kn} = \frac{1}{6}S_{ii}\left(S_{jj}S_{kk} - 3S_{kj}S_{jk}\right) + \frac{1}{3}S_{ji}S_{kj}S_{ik}$

   We have that $\in_{ijk}\in_{lmn} = \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)$

   Thus

$$
\begin{aligned}
\in_{ijk}\in_{lmn} S_{il}S_{jm}S_{kn} &= S_{ii}\left(S_{jj}S_{kk} - S_{kj}S_{jk}\right) - S_{ji}\left(S_{ij}S_{kk} - S_{kj}S_{ik}\right) + S_{ki}\left(S_{ij}S_{jk} - S_{jj}S_{ik}\right) \\
&= S_{ii}\left(S_{jj}S_{kk} - 3S_{kj}S_{jk}\right) + 2S_{ji}S_{kj}S_{ik}
\end{aligned}
$$

   If you are suspicious, you could have mathematica or maple evaluate the result and check that it is indeed equal to the determinant…

[3 POINTS]

7. Let $J = \det(\mathbf{S})$. Show that $\frac{\partial J}{\partial S_{mn}} = JS_{nm}^{-1}$. (this is a very useful result – we often need to differentiate the determinant of a tensor when working with constitutive equations for materials, for example)

We have that

$$J=\frac{1}{6}\in_{ijk}\in_{lmn}S_{il}S_{jm}S_{kn}\Rightarrow\frac{\partial J}{\partial S_{pq}}=\frac{1}{6}\in_{ijk}\in_{lmn}\left(\delta_{ip}\delta_{lq}S_{jm}S_{kn}+S_{il}\delta_{jp}\delta_{mq}S_{kn}+S_{il}S_{jm}\delta_{kp}\delta_{nq}\right)$$

$$=\frac{1}{2}\in_{pjk}\in_{qmn}S_{jm}S_{kn}$$

But recall that $S^{-1}_{ji}=\frac{1}{2\det(\mathbf{S})}\in_{ipq}\in_{jkl}S_{pk}S_{ql}\Rightarrow\frac{\partial J}{\partial S_{pq}}=\det(\mathbf{S})S^{-1}_{qp}$.

[3 POINTS]

8. Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be a Cartesian basis. Vector $\mathbf{u}$ has components $(1,2,0)$ in this basis, while tensors $\mathbf{S}$ and $\mathbf{T}$ have components

$$\mathbf{T}\equiv\begin{bmatrix}1&\sqrt{6}&0\\\sqrt{6}&2&0\\0&0&5\end{bmatrix}\qquad\mathbf{S}\equiv\begin{bmatrix}1&-1&3\\2&4&2\\1&2&6\end{bmatrix}$$

a. Calculate the components of the following vectors and tensors

$$\mathbf{v}=\mathbf{Tu}\quad\mathbf{v}=\mathbf{u}\cdot\mathbf{T}\quad\mathbf{V}=\mathbf{S}+\mathbf{T}\quad\mathbf{V}=\mathbf{S}\cdot\mathbf{T}\quad\mathbf{V}=\mathbf{S}^T$$

These are all straightforward (but tedious) matrix manipulations. The solutions are

$$\mathbf{v}=\begin{bmatrix}1+2\sqrt{6}\\4+\sqrt{6}\\0\end{bmatrix}\qquad\mathbf{v}=[1+2\sqrt{6},4+\sqrt{6},0]\qquad\mathbf{V}=\begin{bmatrix}2&\sqrt{6}-1&3\\2+\sqrt{6}&6&2\\1&2&11\end{bmatrix}$$

$$\mathbf{V}=\begin{bmatrix}1-\sqrt{6}&\sqrt{6}-2&15\\2+4\sqrt{6}&8+2\sqrt{6}&10\\1+2\sqrt{6}&4+\sqrt{6}&30\end{bmatrix}\qquad\mathbf{V}=\begin{bmatrix}1&2&1\\-1&4&2\\3&2&6\end{bmatrix}$$

[3 POINTS]

b. Find the eigenvalues and the components of the eigenvectors of $\mathbf{T}$.

One eigenvalue is 5, with the corresponding eigenvector (0,0,1) by inspection. The others can be computed using the formulas in the notes, or else using mathematica or maple, which gives

$$-1,\ (-\sqrt{3/5},\sqrt{2/5},0)\qquad 4,\ (\sqrt{2/5},\sqrt{3/5},0)$$

[3 POINTS]

c. Denote the three (unit) eigenvectors of $\mathbf{T}$ by $\mathbf{m}_1$, $\mathbf{m}_2$, $\mathbf{m}_3$ (It doesn't matter which eigenvector is which, but be sure to state your choice clearly). Let $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$ be a new Cartesian basis. Write down the components of $\mathbf{T}$ in $\{\mathbf{m}_1,\mathbf{m}_2,\mathbf{m}_3\}$. (Don't make this hard: in the new basis, $\mathbf{T}$ must be diagonal, and the diagonal elements must be the eigenvalues. Do you see why this is the case? You just need to get them in the right order!) Let

$$\mathbf{m}_1=-\sqrt{3/5}\mathbf{e}_1,\sqrt{2/5}\mathbf{e}_2\qquad\mathbf{m}_2=\sqrt{2/5}\mathbf{e}_1+\sqrt{3/5}\mathbf{e}_2\qquad\mathbf{m}_3=\mathbf{e}_3$$

$$\text{Then }\mathbf{T}=\begin{bmatrix}5&0&0\\0&-1&0\\0&0&4\end{bmatrix}$$

[1 POINT]

d. Calculate the components of $\mathbf{S}$ in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$.

This is a pain… The coordinate transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ is

$$\begin{bmatrix} -\sqrt{3/5} & \sqrt{2/5} & 0 \\ \sqrt{6/15} & \sqrt{3/5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation $S_{ij}^{(\mathbf{m})} = Q_{ik} S_{kl}^{(\mathbf{e})} Q_{jl}$ can be computed using Maple or Mathematica, with the result

$$\frac{1}{5}\begin{bmatrix} 11-\sqrt{6} & 3\sqrt{6}+7 & 2\sqrt{10}-3\sqrt{15} \\ 3\sqrt{6}-8 & 14+\sqrt{6} & 3\sqrt{10}+2\sqrt{15} \\ -\sqrt{15}+2\sqrt{10} & \sqrt{10}+2\sqrt{15} & 30 \end{bmatrix}$$

[2 POINTS]

9. Let $\mathbf{S}$ and $\mathbf{T}$ be tensors with components

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 2 & -2 \\ -1 & 4 & 3 \\ 1 & 2 & 6 \end{bmatrix} \qquad \mathbf{S} \equiv \begin{bmatrix} 1 & -1 & 3 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

a. Calculate $\mathbf{S}:\mathbf{T}$ and $\mathbf{S}\cdot\cdot\mathbf{T}$

b. Calculate trace(**S**) and trace(**T**)

This is just busy work again - $\mathbf{S}:\mathbf{T} = 24$ $\quad \mathbf{S}\cdot\cdot\mathbf{T} = 39$ and the traces are 11 and 6.

[2 POINTS]

10. Show that the inner product of two tensors is invariant to a change of basis.

Recall that $S_{ij}^{(\mathbf{m})} = Q_{ik} S_{kp}^{(\mathbf{e})} Q_{jp} \qquad Q_{ik} Q_{jk} = Q_{ki} Q_{kj} = \delta_{ij}$

Therefore $S_{ij}^{(\mathbf{m})} T_{ij}^{(\mathbf{m})} = Q_{ik} S_{kp}^{(\mathbf{e})} Q_{jp} Q_{im} T_{mn}^{(\mathbf{e})} Q_{jn} = Q_{ik} Q_{im} Q_{jn} Q_{jp} S_{kp}^{(\mathbf{e})} T_{mn}^{(\mathbf{e})} = \delta_{mk} \delta_{np} S_{kp}^{(\mathbf{e})} T_{mn}^{(\mathbf{e})} = S_{kp}^{(\mathbf{e})} T_{kp}^{(\mathbf{e})}$

[2 POINTS]

11. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a Cartesian basis. Let $\mathbf{R}$ be a proper orthogonal tensor, and let $\mathbf{m}_i = \mathbf{R}\mathbf{e}_i$

a. Show that $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ is also a Cartesian basis (i.e. show that $\mathbf{m}_i$ are orthogonal unit vectors).

b. Let $R_{ij}^{(\mathbf{e})}, R_{ij}^{(\mathbf{m})}$ denote the components of $\mathbf{R}$ in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, respectively. Show that $R_{ij}^{(\mathbf{e})} = R_{ij}^{(\mathbf{m})}$.

To show that $\mathbf{m}_i$ are orthonormal note that $\mathbf{m}_i \cdot \mathbf{m}_k = (\mathbf{R}\cdot\mathbf{e}_i)\cdot(\mathbf{R}\cdot\mathbf{e}_k) = \mathbf{e}_i \cdot \left(\mathbf{R}^T \cdot \mathbf{R}\right) \cdot \mathbf{e}_k = \mathbf{e}_i \cdot \mathbf{e}_k = \delta_{ik}$

[2 POINTS]

The basis change formula gives $R_{ij}^{(\mathbf{m})} = Q_{ik} R_{kn}^{(\mathbf{e})} Q_{jn}, \qquad Q_{ik} = \mathbf{m}_i \cdot \mathbf{e}_k = (\mathbf{R}\mathbf{e}_i)\cdot\mathbf{e}_k = R_{ki}^{(\mathbf{e})}$. Therefore,

$$R_{ij}^{(\mathbf{m})} = R_{ki}^{(\mathbf{e})} R_{kn}^{(\mathbf{e})} R_{nj}^{(\mathbf{e})} = \delta_{in} R_{nj}^{(\mathbf{e})} = R_{ij}^{(\mathbf{e})}$$

[2 POINTS]