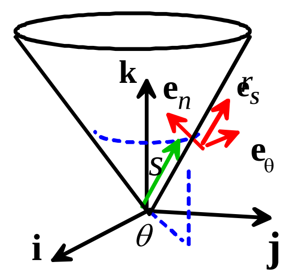
School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 2: Kinematics
Due 12:00 noon Friday February 4th

1. To analyze the deformation of a conical membrane, it is proposed to use a two-dimensional conical-polar coordinate system (s,θ) illustrated in the figure. s denotes the distance of a point on the cone from its apex, and θ is the angle subtended by a radial line and the **i** direction.



(a) Find the coordinate transformation from $\{x_1, x_2, x_3\}$ to s,θ and the inverse

$$x_1 = s\sin\alpha\cos\theta \qquad x_2 = s\sin\alpha\sin\theta \qquad x_3 = s\cos\alpha$$
$$s = \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad \theta = \tan^{-1}(x_2 / x_1)$$

[2 POINTS]

(b) Write down formulas for the three basis vectors

$$\mathbf{e}_s = \frac{\partial \mathbf{r}}{\partial s} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} \qquad \mathbf{e}_n = \mathbf{e}_s \times \mathbf{e}_\theta$$

in the $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ basis.

$$\mathbf{e}_s = \sin\alpha\cos\theta\mathbf{i} + \sin\alpha\sin\theta\mathbf{j} + \cos\alpha\mathbf{k}$$
$$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$
$$\mathbf{e}_n = -\cos\theta\cos\alpha\mathbf{i} - \sin\theta\cos\alpha\mathbf{j} + \sin\alpha\mathbf{k}$$

[2 POINTS]

(c) Hence, determine expressions for

$$\frac{\partial \mathbf{e}_s}{\partial \theta} \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} \qquad \frac{\partial \mathbf{e}_n}{\partial \theta}$$

in terms of $\{\mathbf{e}_s, \mathbf{e}_\theta, \mathbf{e}_n\}$

$$\frac{\partial \mathbf{e}_s}{\partial \theta} = -\sin\alpha\sin\theta\mathbf{i} + \sin\alpha\cos\theta\mathbf{j} = \sin\alpha\mathbf{e}_\theta$$
$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\cos\theta\mathbf{i} - \sin\theta\mathbf{j} = -\sin\alpha\mathbf{e}_s + \cos\alpha\mathbf{e}_n$$
$$\frac{\partial \mathbf{e}_n}{\partial \theta} = \sin\theta\cos\alpha\mathbf{i} - \cos\theta\cos\alpha\mathbf{j} = -\cos\alpha\mathbf{e}_\theta$$

[2 POINTS]

(d) Let $d\mathbf{r} = ds\mathbf{e}_s + s\sin\alpha d\theta\mathbf{e}_\theta$ be an infinitesimal vector that lies in the surface of the cone. Find formulas for $ds, d\theta$ in terms of d**r** and other relevant variables.

$$ds = d\mathbf{r}\cdot\mathbf{e}_s \qquad d\theta = d\mathbf{r}\cdot\mathbf{e}_\theta / (s\sin\alpha)$$

[1 POINT]

(e) Let $\phi(s,\theta)$ be a scalar valued function defined on the surface of the cone. The *surface gradient* of $\nabla_s \phi$ is defined so that $[\nabla_s \phi]\cdot d\mathbf{r} = d\phi$ for all infinitesimal vectors that lie in the surface of the cone. Show that the surface gradient operator is

$$\nabla_s \equiv \left(\frac{\partial}{\partial s}\mathbf{e}_s + \frac{1}{s\sin\alpha}\frac{\partial}{\partial\theta}\mathbf{e}_\theta \right)$$

We have

$$d\phi = \frac{\partial\phi}{\partial s}ds + \frac{\partial\phi}{\partial\theta}d\theta = \left(\frac{\partial}{\partial s}\mathbf{e}_r + \frac{1}{s\sin\alpha}\frac{\partial}{\partial\theta}\mathbf{e}_\theta \right)\phi\cdot d\mathbf{r}$$

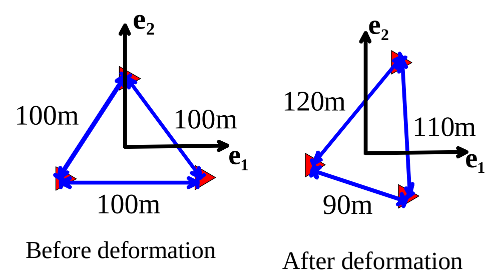
[2 POINTS]

(f) The curvature tensor $\boldsymbol{\kappa}$ of a surface is defined so that $\boldsymbol{\kappa}\cdot d\mathbf{r} = d\mathbf{n}$ gives the difference in normal to the surface $\mathbf{n}$ at two points on the surface separated by an infinitesimal vector $d\mathbf{r}$. Use the solutions to © and (d) to determine the components of $\boldsymbol{\kappa}$ in $\{\mathbf{e}_s, \mathbf{e}_\theta, \mathbf{e}_n\}$

We have that $d\mathbf{e}_n = -\cos\alpha\, d\theta\mathbf{e}_\theta = \mathbf{e}_\theta\left(-\dfrac{\cos\alpha}{s\sin\alpha}\right)\mathbf{e}_\theta\cdot d\mathbf{r}$. Hence $\boldsymbol{\kappa} = -\dfrac{\cos\alpha}{s\sin\alpha}\mathbf{e}_\theta\otimes\mathbf{e}_\theta$

[2 POINTS]

2. To track the deformation in a slowly moving glacier, three survey stations are installed in the shape of an equilateral triangle, spaced 100m apart, as shown in the picture. After a suitable period of time, the spacing between the three stations is measured again, and found to be 90m, 110m and 120m, as shown in the figure. Assuming that the deformation of the glacier is homogeneous over the region spanned by the survey stations, please compute the components of the Lagrange strain tensor associated with this deformation, expressing your answer as components in the basis shown.



Recall that the initial and deformed lengths l, l_0 of a material element parallel to a unit vector $\mathbf{m}$ in the undeformed configuration are related by

$$\frac{l^2 - l_0^2}{2l_0^2} = E_{ij}m_i m_j$$

The three unit vectors parallel to the sides of the triangle are $(1,0,0)$ $(1/2,\sqrt{3}/2,0)$ $(-1/2,\sqrt{3}/2,0)$. Multiplying out the vectors for the three sides of the triangle gives

$$\frac{90^2-100^2}{2.100^2} = E_{11} \qquad \frac{110^2-100^2}{2.100^2} = \frac{E_{11}}{4} + \frac{3E_{22}}{4} - \frac{E_{12}\sqrt{3}}{2}$$

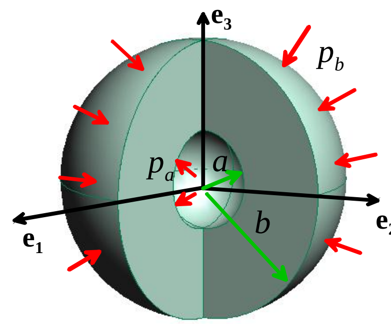
$$\frac{120^2-100^2}{2.100^2} = \frac{E_{11}}{4} + \frac{3E_{22}}{4} + \frac{E_{12}\sqrt{3}}{2}$$

These three equations can be solved for the Lagrange strain components, with the result

$E_{11} = -19/200$ $E_{22} = 149/600$ $E_{12} = 23\sqrt{3}/600$ or $E_{11} = -0.095$ $E_{22} = 0.2483$ $E_{12} = 0.0664$

[5 POINTS]

3. A spherical shell (see the figure) is made from an incompressible material. In its undeformed state, the inner and outer radii of the shell are A, B. After deformation, the new values are a, b. The deformation in the shell can be described (in Cartesian components) by the equation



$$y_i = r\frac{x_i}{R} \qquad r = \left(R^3 + a^3 - A^3\right)^{1/3} \qquad R = \sqrt{x_k x_k}$$

(a) Calculate the components of the deformation gradient tensor

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = \left(R^3 + a^3 - A^3\right)^{-2/3} x_i x_j + \left(R^3 + a^3 - A^3\right)^{1/3}\left(\delta_{ij} - \frac{x_i x_j}{R^3}\right)$$

$$= \delta_{ij}\frac{r}{R} + \left(\frac{R^2}{r^2} - \frac{r}{R}\right)\frac{x_i x_j}{R^2}$$

[2 POINTS]

(b) Verify that the deformation is volume preserving

Since the deformation is radially symmetric, we can compute J along any radial line. Taking $x_1 = R, x_2 = x_3 = 0$, we see that

$$F_{11} = \frac{R^2}{r^2} \qquad F_{22} = F_{33} = \frac{r}{R} \Rightarrow \det(F) = F_{11}F_{22}F_{33} = 1$$

[2 POINTS]

(c) Find the deformed length of an infinitesimal radial line that has initial length l_0, expressed as a function of R

Let $dx_i = l_0\frac{x_i}{R}$. Then $dy_i = F_{ij}dx_j = \left(\delta_{ij}\frac{r}{R} + \left(\frac{R^2}{r^2} - \frac{r}{R}\right)\frac{x_i x_j}{R^2}\right) l_0 \frac{x_j}{R} = l_0 \frac{R^2}{r^2}\frac{x_i}{R}$ and

$\sqrt{dy_i dy_i} = l_0 \frac{R^2}{r^2}$.

[2 POINTS]

(d) Find the deformed length of an infinitesimal circumferential line that has initial length l_0, expressed as a function of R

Since the deformation is volume preserving, $l_\theta^2 l_r = l_0^3 \Rightarrow l_\theta = \sqrt{\frac{l_0^3}{l_r}} = l_0\frac{r}{R}$

[2 POINTS]

(e) Using the results of (c) and (d), write down the principal stretches for the deformation.

If $\mathbf{m}_i$ is a principal stretch direction, the principal stretches λ_i have the property that $l\mathbf{m}_i = \lambda_i l_0 \mathbf{m}_i$ (no sum on *i*). The principal stretch directions are radial and circumferential, by inspection. From (c) and (d), it follows that $\lambda_1 = R^2 / r^2 \quad \lambda_2 = \lambda_3 = r / R$.

[2 POINTS]

(f) Find the *inverse* of the deformation gradient, expressed as a function of y_i. You can do this by inspection, by inverting (a) (not recommended!), or by working out a formula that enables you to calculate x_i in terms of y_i and $r = \sqrt{y_i y_i}$ and differentiating the result. The first is quickest!

Working by inspection,

$$F_{ij}^{-1} = \delta_{ij} \frac{R}{r} + \left(\frac{r^2}{R^2} - \frac{R}{r} \right) \frac{y_i y_j}{r^2}$$

Direct inversion is possible but very tedious. For the third approach, note that

$$x_i = R \frac{y_i}{r} \qquad R = \left(r^3 - a^3 + A^3 \right)^{1/3} \qquad r = \sqrt{y_k y_k}$$

$F_{ij}^{-1} = \partial x_i / \partial y_j$ is then easily computed.

[2 POINTS]

4. Suppose that the spherical shell described in Problem 3 is continuously expanding (visualize a balloon being inflated). The rate of expansion can be characterized by the velocity $v_a = da/dt$ of the surface that lies at *R*=*A* in the undeformed cylinder.

(a) Calculate the velocity field $v_i = dy_i / dt$ in the sphere as a function of x_i

We have that $y_i = r \dfrac{x_i}{R} \qquad r = \left(R^3 + a^3 - A^3 \right)^{1/3} \qquad R = \sqrt{x_k x_k}$

Hence $\dfrac{\partial y_i}{\partial t} = \dfrac{\partial r}{\partial t} \dfrac{x_i}{R} = \dfrac{a^2 v_a}{r^2} \dfrac{x_i}{R}$

[2 POINTS]

(b) Calculate the velocity field as a function of y_i

$$v_i = \frac{a^2 v_a}{r^2} \frac{y_i}{r}$$

[1 POINT]

(c) Calculate the time derivative of the deformation gradient tensor calculated in 2(a)

$$\dot{F}_{ij} = \delta_{ij} \frac{v_a a^2}{R r^2} + \left(-2 \frac{v_a a^2 R^2}{r^5} - \frac{v_a a^2}{R r^2} \right) \frac{x_i x_j}{R^2}$$

[2 POINTS]

(d) Calculate the components of the velocity gradient $L_{ij} = \dfrac{\partial v_i}{\partial y_j}$ by differentiating the result of (b)

$$\frac{\partial v_i}{\partial y_j} = \frac{a^2 v_a}{r^3}\left(\delta_{ij} - 3\frac{y_i y_j}{r^2}\right)$$

[2 POINTS]

(e) Calculate the components of the velocity gradient using the results of © and 2(f)

The other approach is to use

$$\begin{aligned}\frac{\partial v_i}{\partial y_j} = \dot{F}_{ik} F_{kj}^{-1} &= \frac{v_a a^2}{Rr^2}\left[\delta_{ik} - \left(1 + 2\frac{R^3}{r^3}\right)\frac{y_i y_k}{r^2}\right]\left[\delta_{kj}\frac{R}{r} + \left(\frac{r^2}{R^2} - \frac{R}{r}\right)\frac{y_k y_j}{r^2}\right] \\ &= \frac{v_a a^2}{r^3}\left[\delta_{ik} - \left(1 + 2\frac{R^3}{r^3}\right)\frac{y_i y_k}{r^2}\right]\left[\delta_{kj} + \left(\frac{r^3}{R^3} - 1\right)\frac{y_k y_j}{r^2}\right] \\ &= \frac{a^2 v_a}{r^3}\left(\delta_{ij} - 3\frac{y_i y_j}{r^2}\right)\end{aligned}$$

[2 POINTS]

(f) Calculate the stretch rate tensor D_{ij}. Verify that the result represents a volume preserving stretch rate field.

The stretch rate is the symmetric part of $\dfrac{\partial v_i}{\partial y_j}$ - but it is symmetric anyway. So

$$D_{ij} = \frac{a^2 v_a}{r^3}\left(\delta_{ij} - 3\frac{y_i y_j}{r^2}\right)$$

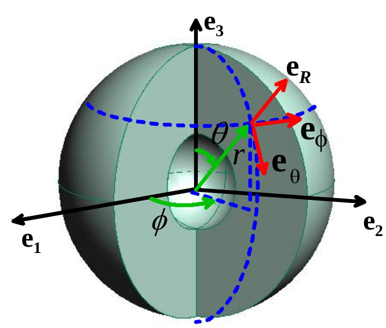
To be volume preserving, $D_{kk} = 0$. It is easy to show that this is indeed satisfied.

[1 POINT]

5. Repeat Problem 3(a), 3(f) and all of 4(b), 4(d), but this time solve the problem using spherical-polar coordinates, using the various formulas for vector and tensor operations given in the notes. In this case, you may assume that a point with position $\mathbf{x} = R\mathbf{e}_R$ in the undeformed solid has position vector

$$\mathbf{y} = \left(R^3 + a^3 - A^3\right)^{1/3}\mathbf{e}_R$$

after deformation.



We need the following results: the gradient operator is $\nabla \equiv \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\theta \frac{1}{R} \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{R\sin\theta} \frac{\partial}{\partial \phi} \right)$

and the derivatives of the basis vectors are

$$\frac{\partial \mathbf{e}_R}{\partial R} = \frac{\partial \mathbf{e}_\theta}{\partial R} = \frac{\partial \mathbf{e}_\phi}{\partial R} = 0 \qquad \frac{\partial \mathbf{e}_R}{\partial \theta} = \mathbf{e}_\theta \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_R \qquad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \mathbf{e}_R}{\partial \phi} = \sin\theta \mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta \mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta \mathbf{e}_R - \cos\theta \mathbf{e}_\theta$$

Then the deformation gradient is: $\mathbf{F} = \nabla(R\mathbf{e}_R) = \frac{R^2}{r^2} \mathbf{e}_R \otimes \mathbf{e}_R + \frac{r}{R} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{r}{R} \mathbf{e}_\phi \otimes \mathbf{e}_\phi$

[2 POINTS]

The inverse can be written down by inspection as

$$\mathbf{F} = \nabla(R\mathbf{e}_R) = \frac{r^2}{R^2} \mathbf{e}_R \otimes \mathbf{e}_R + \frac{R}{r} \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \frac{R}{r} \mathbf{e}_\phi \otimes \mathbf{e}_\phi$$

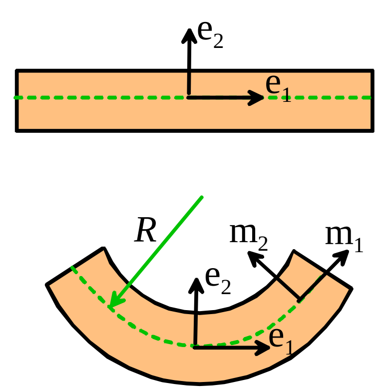
[2 POINTS]

The velocity is $\mathbf{v} = \frac{a^2 v_a}{r^2} \mathbf{e}_R$. **[1 POINT]** The velocity gradient follows as $\nabla_{\mathbf{y}} \mathbf{v}$ where the gradient is taken with respect to deformed coordinates. The gradient operator is the same, however… So

$$\mathbf{L} = \mathbf{D} = \frac{a^2 v_a}{r^3} (-2\mathbf{e}_R \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi)$$

[2 POINTS]

6. An initially straight beam is bent into a circle with radius R as shown in the figure. Material fibers that are perpendicular to the axis of the undeformed beam are assumed to remain perpendicular to the axis after deformation, and the beam's thickness and the length of its axis are assumed to be unchanged. Under these conditions the deformation can be described as



$$y_1 = (R - x_2)\sin(x_1 / R) \qquad y_2 = R - (R - x_2)\cos(x_1 / R)$$

where, as usual $\mathbf{x}$ is the position of a material particle in the undeformed beam, and $\mathbf{y}$ is the position of the same particle after deformation.

(a) Calculate the deformation gradient field in the beam, expressing your answer as a function of x_1, x_2, and as components in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown.

$$\mathbf{F} = \begin{bmatrix} (1 - x_2 / R)\cos(x_1 / R) & -\sin(x_1 / R) \\ (1 - x_2 / R)\sin(x_1 / R) & \cos(x_1 / R) \end{bmatrix}$$

[2 POINTS]

(b) Calculate the Lagrange strain field in the beam.

$$\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2 = \frac{1}{2}\begin{bmatrix} -(x_2/R)(2 - x_2/R) & 0 \\ 0 & 0 \end{bmatrix}$$

[1 POINT]

(c) Calculate the infinitesimal strain field in the beam.

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \begin{bmatrix} (1 - x_2/R)\cos x_1/R - 1 & \{-(x_2/R)\sin(x_1/R)\}/2 \\ \{-(x_2/R)\sin(x_1/R)\}/2 & \cos(x_1/R) - 1 \end{bmatrix}$$

[2 POINTS]

(d) Compare the values of Lagrange strain and infinitesimal strain for two points that lie at $(x_1 = 0, x_2 = h)$ and $(x_1 = L, x_2 = 0)$. Explain briefly the physical origin of the difference between the two strain measures at each point. Recommend maximum allowable values of *h/R* and *L/R* for use of the infinitesimal strain measure in modeling beam deflections.

At the first point, $\mathbf{E} - \boldsymbol{\varepsilon} = \begin{bmatrix} -h^2/2R^2 & 0 \\ 0 & 0 \end{bmatrix}$

At the second $\mathbf{E} - \boldsymbol{\varepsilon} = \begin{bmatrix} 1 - \cos(L/R) & 0 \\ 0 & 1 - \cos(L/R) \end{bmatrix}$

The difference between the two measures at the first point is because the Lagrange strain measure quantifies the change in squared length:

$$\varepsilon_L = \frac{l^2 - l_0^2}{2l_0^2}$$

The infinitesimal strain, on the other hand, gives $\varepsilon = \frac{l - l_0}{l_0}$. The two are equal for small strains, but the quadratic term becomes important for large strains. *h/R<10* is usually a safe range.

At the second point, the difference is a consequence of the rotation of the beam – the incorrect strain predicted by the infinitesimal rotation tensor is the difference between the actual length of the beam and its horizontal projection at the end. $R/L > 10$ is a safe range to avoid this error.

[3 POINTS]

(e) Calculate the deformed length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_1$ in the undeformed beam. Express your answer as a function of x_2.

From the definition of Lagrange strain, we get

$$l = l_0\sqrt{1 - \frac{2x_2}{R}\left(1 - \frac{x_2}{2R}\right)} = l_0(1 - x_2/R)$$

[**1 POINT]**

(f) Calculate the change in length of an infinitesimal material fiber that has length l_0 and orientation $\mathbf{e}_2$ in the undeformed beam.

The length is unchanged - $l = l_0$

[1 POINT]

(g) Show that the two material fibers described in (3) and (f) remain mutually perpendicular after deformation. Is this true for *all* material fibers that are mutually perpendicular in the undeformed solid?

$$\mathbf{F}\cdot\mathbf{e}_1 = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) \\ (1-x_2/R)\sin(x_1/R) \end{bmatrix}$$
$$\mathbf{F}\cdot\mathbf{e}_2 = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(x_1/R) \\ +\cos(x_1/R) \end{bmatrix}$$
$$\left(\mathbf{F}\cdot\mathbf{e}_2\right)\cdot\left(\mathbf{F}\cdot\mathbf{e}_1\right) = 0$$

The result is not true for arbitrary fibers – for example

$$\mathbf{F}\cdot(\mathbf{e}_1+\mathbf{e}_2)/\sqrt{2} = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} (1-x_2/R)\cos(x_1/R)-\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R)+\cos(x_1/R) \end{bmatrix}$$
$$\mathbf{F}\cdot(\mathbf{e}_2-\mathbf{e}_1)/\sqrt{2} = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -(1-x_2/R)\cos(x_1/R)-\sin(x_1/R) \\ -(1-x_2/R)\sin(x_1/R)+\cos(x_1/R) \end{bmatrix}$$
$$\left(\mathbf{F}\cdot(\mathbf{e}_2-\mathbf{e}_1)/\sqrt{2}\right)\cdot\left(\mathbf{F}\cdot(\mathbf{e}_1+\mathbf{e}_2)/\sqrt{2}\right) = \left[1-(1-x_2/R)\right]/2 \neq 0$$

[2 POINTS]

(h) Find the components in the basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of the Left and Right stretch tensors $\mathbf{U}$ and $\mathbf{V}$ as well as the rotation tensor $\mathbf{R}$ for this deformation. You should be able to write down $\mathbf{U}$ and $\mathbf{R}$ by inspection, without needing to wade through the laborious general process. The results can then be used to calculate $\mathbf{V}$.

By inspection, the principal stretch directions are parallel to $\mathbf{e}_1,\mathbf{e}_2$. The rotation tensor is the mapping of $\mathbf{e}_1,\mathbf{e}_2$ to $\mathbf{m}_1,\mathbf{m}_2$ Thus

$$\mathbf{U} = \begin{bmatrix} (1-x_2/R) & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos(x_1/R) & -\sin(x_1/R) \\ \sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$$

$\mathbf{V}$ can then be calculated as

$$\mathbf{V} = \mathbf{F}\mathbf{R}^T = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}\begin{bmatrix} \cos(x_1/R) & \sin(x_1/R) \\ -\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$$
$$= \begin{bmatrix} (1-x_2/R)\cos^2(x_1/R)+\sin^2(x_1/R) & (-x_2/R)\sin(x_1/R)\cos(x_1/R) \\ (-x_2/R)\sin(x_1/R)\cos(x_1/R) & (1-x_2/R)\sin^2(x_1/R)+\cos^2(x_1/R) \end{bmatrix}$$

[3 POINTS]

(i) Find the principal directions of $\mathbf{U}$ as well as the principal stretches. You should be able to write these down without doing any tedious calculations.

This is trivial – the principal directions are just $\mathbf{e}_1, \mathbf{e}_2$; the principal stretches are $(1 - x_2 / R)$,1

[1 POINT]

(j) Let $\{\mathbf{m}_1, \mathbf{m}_2\}$ be a basis in which $\mathbf{m}_1$ is parallel to the axis of the deformed beam, as shown in the figure. Write down the components of each of the unit vectors $\mathbf{m}_i$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Hence, compute the transformation matrix $Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$ that is used to transform tensor components from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{m}_1, \mathbf{m}_2\}$.

$$\mathbf{Q} = \begin{bmatrix} \cos(x_1 / R) & \sin(x_1 / R) \\ -\sin(x_1 / R) & \cos(x_1 / R) \end{bmatrix}$$

[2 POINTS]

(k) Find the components of the deformation gradient tensor, Lagrange strain tensor, as well as $\mathbf{U}$ $\mathbf{V}$ and $\mathbf{R}$ in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. It is best to do these with a symbolic manipulation program.

These calculations can be done quickly with Maple or Mathematica. We get

$$\mathbf{F} = \begin{bmatrix} (1 - x_2 / R)\cos(x_1 / R) & -(1 - x_2 / R)\sin(x_1 / R) \\ \sin(x_1 / R) & \cos(x_1 / R) \end{bmatrix}$$

The components of $\mathbf{V}$ in $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ are equal to the components of $\mathbf{U}$ in $\{\mathbf{e}_1, \mathbf{e}_2\}$, the components of **U in** $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ are the same those of $\mathbf{V}$ in $\{\mathbf{e}_1, \mathbf{e}_2\}$ and as you showed in HW1 the components of $\mathbf{R}$ are the same in both bases. It is not hard to show that these are general properties of these tensors….

[2 POINTS]

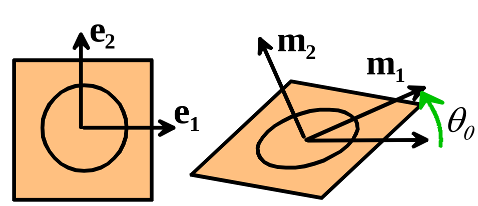
(l) Find the principal directions of $\mathbf{V}$ expressed as components in the basis $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. Again, you should be able to simply write down this result.

This is trivial – the principal directions are just $\mathbf{m}_1, \mathbf{m}_2$; the principal stretches are $(1 - x_2 / R)$,1

[1 POINT]

7. A sheet of material is subjected to a two dimensional homogeneous deformation of the form

$$y_1 = A_{11}x_1 + A_{12}x_2 \qquad y_2 = A_{21}x_1 + A_{22}x_2$$



where A_{ij} are constants. Suppose that a circle of unit radius is drawn on the undeformed sheet. This circle is distorted to a smooth curve on the deformed sheet. Show that the distorted circle is an ellipse, with semi-axes that are parallel to the principal directions of the left stretch tensor **V,** and that the lengths of the semi-axes of the ellipse are equal to the principal stretches for the deformation. There are many different ways to approach this calculation – some are very involved. The simplest way is probably to assume that the principal directions of $\mathbf{V}$ subtend an angle θ_0 to the $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis as shown in the figure, write the polar decomposition $\mathbf{A} = \mathbf{V} \cdot \mathbf{R}$ in terms of principal stretches λ_1, λ_2 and θ_0, and then show that $\mathbf{y} = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{x}$ (where $\mathbf{x}$ is on the unit circle) describes an ellipse.

Note that the rotation $\mathbf{R}$ does not distort the circle at all. To see this, let $\mathbf{z} = \mathbf{R}\mathbf{x}$ and note that $\mathbf{x} \cdot \mathbf{x} = \mathbf{z} \cdot \mathbf{z} = a^2$ where a is the radius of the circle.

The right stretch tensor can be expressed as $\mathbf{V} = \lambda_1 \mathbf{m}_1 \otimes \mathbf{m}_1 + \lambda_2 \mathbf{m}_2 \otimes \mathbf{m}_2$. If we let $\mathbf{z} = \mathbf{R}\mathbf{x}$ and express $\mathbf{z}$ as $\mathbf{z} = z_1 \mathbf{m}_1 + z_2 \mathbf{m}_2$ then $\mathbf{V}\mathbf{z} = \lambda_1 z_1 \mathbf{m}_1 + \lambda_2 z_2 \mathbf{m}_2$. Therefore $\dfrac{y_1^2}{\lambda_1^2} + \dfrac{y_2^2}{\lambda_2^2} = z_1^2 + z_2^2 = a^2$. This is the equation of an ellipse centered at the origin with semiaxes $\lambda_1 a, \lambda_2 a$.

[5 POINTS]

8. The center of mass and the mass moment of inertia tensor in the reference and deformed configurations of a solid are (by definition)

$$r_i^{c0} = \frac{1}{M} \int_{V_0} x_i \rho_0 dV_0 \qquad I_{ij}^{c0} = \int_{V_0} \left(x_i - r_i^{c0}\right)\left(x_j - r_j^{c0}\right) \rho_0 dV_0$$

$$r_i^{c} = \frac{1}{M} \int_{V} y_i \rho dV \qquad I_{ij}^{c} = \int_{V} \left(y_i - r_i^{c}\right)\left(y_j - r_j^{c}\right) \rho dV$$

where ρ_0, ρ are the mass density of the solid in the reference and deformed configurations, $\mathbf{x}, \mathbf{y}$ are the positions of material particles in the reference and deformed configurations, and M is the total mass.

Suppose that a solid is subjected to a homogeneous deformation

$$y_i = A_{ik} x_k + c_i$$

where A_{ij} and c_i are constants.

(a) Find formulas for r_{ic}, I_{ij}^{C} in terms of r_i^{c0} I_{ij}^{c0}, A_{ij} and c_i.

$$r_i^c = \frac{1}{M} \int_V (A_{ik} x_k + c_i) \rho dV \;\; = \frac{1}{M} A_{ik} \int_{V_0} (x_k) \rho_0 dV \;\; + \frac{1}{M} \int_{V_0} (c_i) \rho_0 dV \;\; = A_{ik} r_k^{c0} + c_i$$

$$I_{ij}^c = \int_V \left(y_i - r_i^c\right)\left(y_j - r_j^c\right) \rho dV = \int_V \left(A_{ik} x_k + c_i - r_i^c\right)\left(A_{jl} x_l + c_j - r_j^c\right) \rho dV$$

$$= \int_{V_0} \left(A_{ik}(x_k - r_k^{c0})\right)\left(A_{jl}(x_l - r_l^{c0})\right) \rho_0 dV = A_{ik} I_{kl}^{0} A_{jl}$$

[2 POINTS]

(b) Suppose that A_{ij} is a rigid rotation (this means $A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij}$. Use the solution to (a) to show that the time derivative of I_{ij} can be expressed as

$$\frac{dI_{ij}^{c}}{dt} = W_{ik}I_{kj}^{c} - I_{ik}^{c}W_{kj}$$

where $W_{ik} = \dfrac{dA_{ik}}{dt}A_{jk}$ is the spin tensor.

We can write

$$\begin{aligned}
\frac{dI_{ij}^{c}}{dt} &= \frac{d}{dt}\left(A_{ik}I_{kl}^{0}A_{jl}\right) = \frac{dA_{ik}}{dt}I_{kl}^{0}A_{jl} + A_{ik}I_{kl}^{0}\frac{dA_{jl}}{dt} \\
&= \frac{dA_{ip}}{dt}A_{qp}A_{qk}I_{kl}^{0}A_{jl} + A_{ik}I_{kp}^{0}A_{qp}A_{ql}\frac{dA_{jl}}{dt} \\
&= \frac{dA_{ip}}{dt}A_{qp}I_{qj}^{c} + I_{iq}^{c}A_{qj}\frac{dA_{jl}}{dt} \\
&= W_{iq}I_{qj}^{c} - I_{iq}^{c}W_{ql}
\end{aligned}$$

Here, we have used the result that $A_{ik}A_{jk} = \delta_{ij} \Rightarrow \dfrac{dA_{ik}}{dt}A_{jk} + A_{ik}\dfrac{dA_{jk}}{dt} = 0 \Rightarrow A_{ik}\dfrac{dA_{jk}}{dt} = -W_{ij}$

[2 POINTS]

(c) Suppose that a rigid body rotates with angular velocity ω_k and therefore has angular momentum

$$h_i = I_{ij}^{c}\omega_j$$

Use (b) to show that the time derivative of the angular momentum is

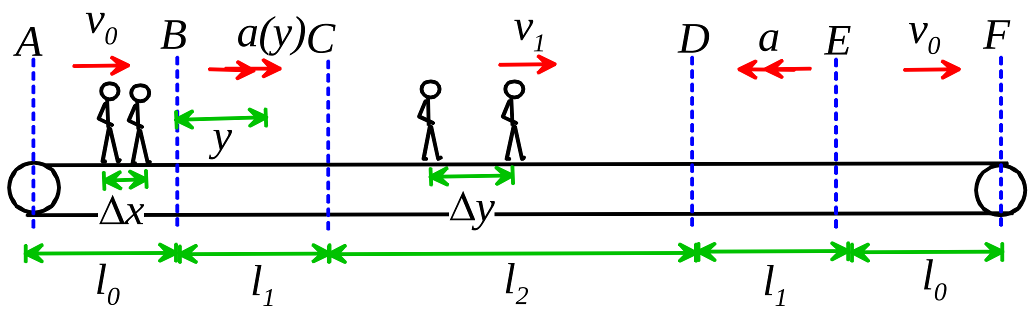
$$\frac{dh_i}{dt} = I_{ij}\frac{d\omega_j}{dt} + \in_{ijk}\omega_j I_{kl}\omega_l$$

Taking the time derivative gives

$$\frac{dh_i}{dt} = \frac{dI_{ij}^{c}}{dt}\omega_j + I_{ij}^{c}\frac{d\omega_j}{dt} = \left(W_{iq}I_{qj}^{c} - I_{iq}^{c}W_{qj}\right)\omega_j + I_{ij}^{c}\frac{d\omega_j}{dt}$$

Recall also that ω_k is the dual vector of W_{ik} . This means that for any vector u_i , $W_{ji}u_i = \varepsilon_{jki}\omega_k u_i$. Substituting this into the preceding result, and noting that $W_{ji}\omega_i = 0$ (the cross product of a vector with itself) gives the required formula. Note that the solution is the standard formula from 3D rigid body dynamics.

[2 POINTS]



9. The figure shows a design for a high-speed moving walkway (see http://www.jfe-steel.co.jp/archives/en/nkk_giho/84/pdf/84_10.pdf for a detailed description of this general type of design, or **http://www.youtube.com/watch?v=uwHer1RrYg8** for a movie of such a walkway in action). A passenger standing on the walkway passes through five regions:

(i) between A and B she moves at constant speed v_0;

(ii) between B and C she accelerates (with an acceleration to be specified below);

(iii) between C and D she moves with constant (high) speed v_1; and

(iv) between D and E she decelerates

(v) between E and F she travels at speed v_0 again.

In this problem we will just focus on portion (ii) of the motion – i.e. between *B* and *C*.

(a) Suppose that the walkway is designed so that the velocity varies linearly with distance between *B* and *C*. Assume that a person walks with speed *w* relative to the moving walkway. Determine her acceleration as a function of distance *y* from *B*, and also as a function of time after passing the point *B*. Find a formula for the maximum value of the acceleration, and identify the point where it occurs.

The velocity of the walkway is $v = v_0 + (v_1 - v_0)y / l_1$. The velocity of the person at a distance *y* from B follows as

$\frac{dy}{dt} = v + w$. The acceleration is $\frac{d^2y}{dt^2} = \frac{dv}{dy}\frac{dy}{dt} = \frac{v_1 - v_0}{l_1}\left((v_0 + w) + (v_1 - v_0)\frac{y}{l_1}\right)$

To find the acceleration as a function of time we must find *y(t)*, which follows as

$$\int_0^y \frac{dy}{v_0 + w + (v_1 - v_0)y / l_1} = \int_0^t dt$$

$$\Rightarrow (v_1 - v_0)y / l_1 = (v_0 + w)\left\{\exp\left[(v_1 - v_0)t / l_1\right] - 1\right\}$$

$$\Rightarrow a(t) = \frac{v_1 - v_0}{l_1}(v_0 + w)\exp\left[(v_1 - v_0)t / l_1\right]$$

The maximum acceleration occurs at $y = l_1$ and has value $a_{\max} == \frac{v_1 - v_0}{l_1}\left((v_1 + w)\right)$

[3 POINTS]

(b) Suppose the walkway is designed instead so that a person standing on the track has constant acceleration *a*. Calculate the required velocity distribution *v(y)* as a function of distance *y* from B, and determine the acceleration of the person walking along the accelerating walkway as a function of *y* and also a function of *t*.

If the acceleration is constant, then $v=\sqrt{v_1^2+2ay}$ (straight line motion formulas)

Then $\dfrac{d^2y}{dt^2}=\dfrac{dv}{dy}\dfrac{dy}{dt}=\dfrac{a}{\sqrt{v_1^2+2ay}}\left(\sqrt{v_1^2+2ay}+w\right)$

In this case

$$\frac{dy}{dt}=v+w=w+\sqrt{v_1^2+2ay}\Rightarrow\int_0^y\frac{dy}{w+\sqrt{v_1^2+2ay}}=t$$

$$\Rightarrow\frac{\sqrt{v_1^2+2ay}-v_1}{a}-\frac{w}{v}\log\left(\frac{w+\sqrt{v_1^2+2ay}}{w+v_1}\right)=t$$

$$\Rightarrow\beta-1+\alpha\log\frac{1+\beta}{1+\alpha}=\tau\qquad\beta=\sqrt{1+2ay/v_1}\qquad\alpha=w/v\qquad\tau=at/v_1$$

$$\beta=\alpha P\left(\frac{(1+\alpha)\exp((2+\tau)/\alpha)}{\alpha}\right)-1$$

where $w=P(z)$ is the principal root of $z=w\exp(w)$ (the ProductLog function in Mathematica)

Hence

$$\frac{d^2y}{dt^2}=\frac{a}{2}\frac{d^2(\beta^2-1)}{d\tau^2}=\frac{aP(z)\left(\alpha P(z)^2+2\alpha P(z)-1\right)}{\alpha\left(1+P(z)\right)^3}\qquad z=\frac{(1+\alpha)\exp((2+\tau)/\alpha)}{\alpha}$$

[3 POINTS]