School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 3: Kinetics
Due 12:00 noon Friday February 4th

1. For the Cauchy stress tensor with components

$$\begin{bmatrix} 100 & 250 & 0 \\ 250 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

(MPa) compute

(a) The traction vector acting on an internal material plane with normal $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$

The traction follows as $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 100 & 250 & 0 \\ 250 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -150 \\ 50 \\ 0 \end{bmatrix}$

(b) The principal stresses

One principal is 300 by inspection. MATLAB gives the others -105MPa and 405MPa.

(c) The hydrostatic stress

The hydrostatic stress is one third of the trace of the stress tensor: 200 MPa

(d) The deviatoric stress tensor

The deviatoric stress is $\sigma_{ij}^{D} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3 = \begin{bmatrix} -100 & 250 & 0 \\ 250 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix}$

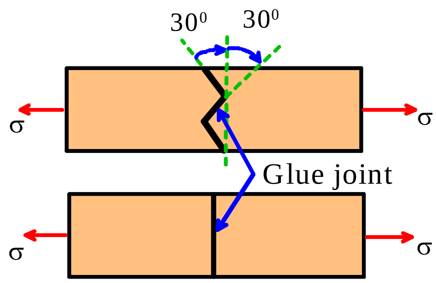
(e) The Von-Mises equivalent stress

The V-M stress is $\sigma_e = \sqrt{\frac{3}{2}\sigma_{ij}^{D}\sigma_{ij}^{D}} = 466MPa$

2. The figure shows two designs for a glue joint. The glue will fail if the stress acting normal to the joint exceeds 60 MPa, or if the shear stress acting parallel to the plane of the joint exceeds 300 MPa.

(a) Calculate the normal and shear stress acting on each joint, in terms of the applied stress σ

(b) Hence, calculate the value of σ that will cause each joint to fail.



Taking $\mathbf{e}_1$ to be horizontal, the normal vectors to the two joints are $\mathbf{n}_1 = \cos(30)\mathbf{e}_1 + \sin(30)\mathbf{e}_2 \quad \mathbf{n}_2 = \mathbf{e}_1$. The stress tensor is $\boldsymbol{\sigma} = \sigma \mathbf{e}_1 \otimes \mathbf{e}_1$. The tractions follow as $\mathbf{t}_1 = \sigma \cos(30)\mathbf{e}_1 \quad \mathbf{t}_2 = \sigma \mathbf{e}_1$, and hence the normal and tangential components of traction are

$t_n = \mathbf{t}_1 \cdot \mathbf{n}_1 = \sigma \cos^2(30) \quad t_t = \left|\mathbf{t}_1 - t_n \mathbf{n}_1\right| = \sigma \cos(30)\left|\left(\mathbf{e}_1 - \cos(30)(\cos(30)\mathbf{e}_1 + \sin(30)\mathbf{e}_2\right)\right| = \sigma \sin(30)\cos(30)$

For the second joint $t_n = \sigma \ t_t = 0$.

3. An internal surface plane that makes equal angles with each of the three principal stress directions is known as the *octahedral plane*. Show that the normal component of stress acting on this plane is $I_1/3$, and that the magnitude of the shear traction acting on the plane is

$$\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2} = \sqrt{-2I_2'/3}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ are the three principal stresses, and I_2' is the second invariant of the *deviatoric* stress tensor $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$

Denote the principal stresses and directions by $\{\sigma_1, \sigma_2, \sigma_3\}, \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ - we can then write

$$\boldsymbol{\sigma} = \sigma_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \sigma_3 \mathbf{n}_3 \otimes \mathbf{n}_3$$

If the normal to the octahedral plane makes an equal angle to $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, then $\mathbf{m} = (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)/\sqrt{3}$

The traction on this plane follows as $\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} = \left(\sigma_1 \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 + \sigma_3 \mathbf{n}_3\right)/\sqrt{3}$

The normal traction is $t_n = \mathbf{t} \cdot \mathbf{m} = (\sigma_1 + \sigma_2 + \sigma_3)/3$. This is $trace(\boldsymbol{\sigma})/3$.

The shear traction follows as

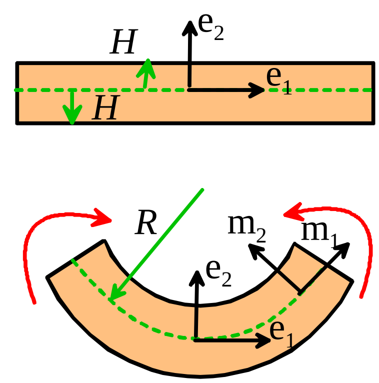
$$\begin{aligned} t_t = \left|\mathbf{t} - t_n \mathbf{m}\right| &= \left|\left(\sigma_1 \mathbf{n}_1 + \sigma_2 \mathbf{n}_2 + \sigma_3 \mathbf{n}_3\right)/\sqrt{3} - (\sigma_1 + \sigma_2 + \sigma_3)(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)/3\sqrt{3}\right| \\ &= \left|(2\sigma_1 - \sigma_2 - \sigma_3)\mathbf{n}_1/3\sqrt{3} + (2\sigma_2 - \sigma_1 - \sigma_3)\mathbf{n}_2/3\sqrt{3} + (2\sigma_3 - \sigma_1 - \sigma_2)\mathbf{n}_3/3\sqrt{3}\right| \\ &= \sqrt{3\left[(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2\right]/27} \end{aligned}$$

Finally note also that $I_2' = -(\sigma_{ij} - \sigma_{kk}\delta_{ij}/3)(\sigma_{ij} - \sigma_{nn}\delta_{ij}/3) = \sigma_{kk}\sigma_{nn}/3 - \sigma_{ij}\sigma_{ij}$. Substituting the principal stresses into this expression gives the required result.

4. In this problem we consider further the beam bending calculation discussed in HW2. Suppose that the beam is made from a material in which the Material Stress tensor is related to the Lagrange strain tensor by

$$\Sigma_{ij} = 2\mu E_{ij}$$

(this can be regarded as representing an elastic material with zero Poisson's ratio and shear modulus μ)



(a) Calculate the distribution of material stress in the bar, expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

From HW2, we have the Lagrange Strain $\mathbf{E} = (\mathbf{F}^T\mathbf{F} - \mathbf{I})/2 = \frac{1}{2}\begin{bmatrix} -(x_2/R)(2 - x_2/R) & 0 \\ 0 & 0 \end{bmatrix}$. The stress follows as

$$\mathbf{\Sigma} = \mu \begin{bmatrix} -(x_2/R)(2 - x_2/R) & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Calculate the distribution of nominal stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

The nominal stress is $\mathbf{S} = \mathbf{\Sigma}\mathbf{F}^T$

Recall that $\mathbf{F} = \begin{bmatrix} (1 - x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1 - x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$, and so

$$\mathbf{S} = \mu \begin{bmatrix} -(x_2/R)(2 - x_2/R) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (1 - x_2/R)\cos(x_1/R) & (1 - x_2/R)\sin(x_1/R) \\ -\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$$

$$= \mu \begin{bmatrix} -(x_2/R)(2 - x_2/R)(1 - x_2/R)\cos(x_1/R) & -(x_2/R)(2 - x_2/R)(1 - x_2/R)\sin(x_1/R) \\ 0 & 0 \end{bmatrix}$$

(c) Calculate the distribution of Cauchy stress in the bar expressing your answer as components in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis

The Cauchy stress is

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\mathbf{S} = (x_2/R)(2 - x_2/R) \begin{bmatrix} (1 - x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1 - x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix} \begin{bmatrix} -\cos(x_1/R) & -\sin(x_1/R) \\ 0 & 0 \end{bmatrix}$$

$$= \mu(x_2/R)(2 - x_2/R)(1 - x_2/R) \begin{bmatrix} -\cos^2(x_1/R) & -\sin(x_1/R)\cos(x_1/R) \\ -\sin(x_1/R)\cos(x_1/R) & -\sin^2(x_1/R) \end{bmatrix}$$

(d) Repeat (a)-(c) but express the stresses as components in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis

The basis change matrix is

$$Q = \begin{bmatrix} \cos(x_1/R) & \sin(x_1/R) \\ -\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$$

Thus $\sigma^{(\mathbf{m})} = Q\sigma Q^T = \mu(x_2/R)(2-x_2/R)\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

and similarly

$$\mathbf{S} = (x_2/R)(2-x_2/R)(1-x_2/R)\begin{bmatrix} -\cos^2(x_1/R) & 0 \\ -\sin(x_1/R)\cos(x_1/R) & 0 \end{bmatrix}$$

$$\mathbf{\Sigma} = \mu(x_2/R)(2-x_2/R)\begin{bmatrix} -\cos^2(x_1/R) & \sin(x_1/R)\cos(x_1/R) \\ \sin(x_1/R)\cos(x_1/R) & -\sin^2(x_1/R) \end{bmatrix}$$

(e) Calculate the distribution of traction on a surface in the beam that has normal $\mathbf{e}_1$ in the undeformed beam. Give expressions for the tractions in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

The traction can be computed from the nominal stress as

$$\mathbf{t} = \mathbf{n}\cdot\mathbf{S} = \mu(x_2/R)(2-x_2/R)(1-x_2/R)\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -\cos(x_1/R) & -\sin(x_1/R) \\ 0 & 0 \end{bmatrix}$$

$$= -\mu(x_2/R)(2-x_2/R)(1-x_2/R)[\cos(x_1/R) \quad \sin(x_1/R)]$$

Note that this is in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ basis. In the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis the answer is simply

$$\mathbf{t} = -\mu(x_2/R)(2-x_2/R)(1-x_2/R)[1 \quad 0]$$

(f) Show that the surfaces of the beam that have positions $x_2 = \pm h/2$ in the undeformed beam are traction free after deformation

The traction on a surface with normal in the $\mathbf{e}_2$ direction is

$$\mathbf{t} = \mathbf{n}\cdot\mathbf{S} = \mu(x_2/R)(2-x_2/R)(1-x_2/R)\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -\cos(x_1/R) & -\sin(x_1/R) \\ 0 & 0 \end{bmatrix}$$

$$= [0 \quad 0]$$

(g) Calculate the resultant moment (per unit out of plane distance) acting on the ends of the beam.

The resultant moment is

$$\mathbf{M} = \int_{-H}^{H} (\mathbf{r}-\mathbf{r}_0)\times\mathbf{t} = \int_{-H}^{H} -\mu(x_2/R)(2-x_2/R)(1-x_2/R)x_2(\mathbf{m}_2\times\mathbf{m}_1)dx_2$$

$$= \frac{2}{15}\frac{H^3}{R^3}\left(3H^2 + 10R^2\right)$$

5. **5.** A solid is subjected to some loading that induces a Cauchy stress $\sigma_{ij}^{(0)}$ at some point in the solid. The solid and the loading frame are then rotated together so that the entire solid (as well as the loading frame) is subjected to a rigid rotation R_{ij}**.** This causes the components of the Cauchy stress tensor to change to new values $\sigma_{ij}^{(1)}$. The goal of this problem is to calculate a formula relating $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.

   a. Let $n_i^{(0)}$ be a unit vector normal to an internal material plane in the solid before rotation. After rotation, this vector (which rotates with the solid) is $n_i^{(1)}$. Write down the formula relating $n_i^{(0)}$ and $n_i^{(1)}$

   $$n_i^{(1)} = R_{ij} n_j^{(0)}$$

   b. Let $T_i^{(0)}$ be the internal traction vector that acts on a material plane with normal $n_i^{(0)}$ in the solid before application of the rigid rotation. Let $T_i^{(1)}$ be the traction acting on the same material plane after rotation. Write down the formula relating $T_i^{(0)}$ and $T_i^{(1)}$

   $$T_i^{(1)} = R_{ij} T_j^{(0)}$$

   c. Finally, using the definition of Cauchy stress, find the relationship between $\sigma_{ij}^{(0)}$, $\sigma_{ij}^{(1)}$ and R_{ij}.

   $$T_i^{(1)} = n_j^{(1)} \sigma_{ji}^{(1)} \Rightarrow R_{ij} T_j^{(0)} = R_{kl} n_l^{(0)} \sigma_{ki}^{(1)}$$
   $$\Rightarrow R_{in} R_{ij} T_j^{(0)} = T_n^{(0)} = n_l^{(0)} R_{kl} \sigma_{ki}^{(1)} R_{in}$$

   Hence $\sigma_{ln}^{(0)} = R_{kl} \sigma_{ki}^{(1)} R_{in} \Rightarrow \sigma_{ki}^{(1)} = R_{kl} \sigma_{ln}^{(0)} R_{in}$

6. **6.** Repeat problem 5, but instead, calculate a relationship between the components of Nominal stress $S_{ij}^{(0)}$ and $S_{ij}^{(1)}$ before and after the rigid rotation.

For nominal stress, letting N_i denote the normal to the reference configuration, we have

$$T_i^{(1)} = N_j S_{ji}^{(1)} \Rightarrow R_{ij} T_j^{(0)} = N_j S_{ji}^{(1)}$$
$$\Rightarrow R_{in} R_{ij} T_j^{(0)} = T_n^{(0)} = N_j S_{ji}^{(1)} R_{in}$$

Thus $\sigma_{ln}^{(0)} = R_{kl} \sigma_{ki}^{(1)} R_{in} \Rightarrow \sigma_{ki}^{(1)} = R_{kl} \sigma_{ln}^{(0)} R_{in}$

7. **7.** Repeat problem 5, but instead, calculate a relationship between the components of material stress $\Sigma_{ij}^{(0)}$ and $\Sigma_{ij}^{(1)}$ before and after the rigid rotation.

The material stress is unaffected by the rigid rotation – you can see this from the physical interpretation of material stress, or else use the formula relating material stress to Cauchy stress, and note that the deformation gradient for stretch followed by rotation is related to that associated with just the stretch by $\mathbf{F}^{(1)} = \mathbf{R}\mathbf{F}^{(0)}$. Substituting this and the results of (5) into the formula $\mathbf{\Sigma}_{ij} = JF_{ik}^{-1}\sigma_{kl}F_{jl}^{-1}$ and simplifying the result shows that $\Sigma_{ij}^{(0)} = \Sigma_{ij}^{(1)}$

8. One constitutive model for metallic glass (Anand and Su, J. Mech Phys. Solids **53** 1362 (2005)) assumes that plastic flow in the glass takes place by shearing on planes that are oriented at an angle θ to the principal stress directions, calculated as follows. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be unit vectors parallel to the three principal stresses (with $\sigma_1 > \sigma_2 > \sigma_3$) and suppose that shearing takes place on a plane with normal $\mathbf{m}$, with shearing direction (tangent to the plane) $\mathbf{s}$. let $\tau = \mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{s}$ and $p_n = -\mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{m}$ denote the resolved shear stress and (compressive) normal stress acting on the shear plane. The constitutive model assumes that shearing in the $\{\mathbf{e}_1, \mathbf{e}_3\}$ plane occurs on the plane for which

$$f(\theta) = \tau(\theta) - \mu p(\theta)$$

is a maximum with respect to θ. Here μ is a material property known as the 'internal friction coefficient.' Their paper (eq 71) states that 'it is easily shown that $f(\theta)$ is a maximum for

$$\theta = \left\{ \frac{\pi}{4} \pm \frac{\phi}{2} \right\} \qquad \phi = \tan^{-1} \mu$$

Derive this result.

The stress can be expressed as

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_3 \end{bmatrix} \qquad \mathbf{s} = [\cos\theta, \sin\theta] \qquad \mathbf{m} = [-\sin\theta, \cos\theta]$$

The normal pressure and shear traction follow as

$$\tau = (\sigma_3 - \sigma_1)\sin\theta\cos\theta$$

$$p_n = -\sigma_1 \sin^2\theta - \sigma_3 \cos^2\theta$$

$$\tau - \mu p_n = (\sigma_3 - \sigma_1)\sin\theta\cos\theta + \sigma_1 \sin^2\theta + \sigma_3 \cos^2\theta$$

Differentiation leads to the following equation

$$(\sigma_3 - \sigma_1)(\cos 2\theta - \mu \sin 2\theta) = 0$$

This can be rearranged into the form stated.