School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 4: Balance laws, work and energy, virtual work
Due Friday Oct 26th

1. Show that the local mass balance equation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$

can be re-written in spatial form as

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

2. The stress field

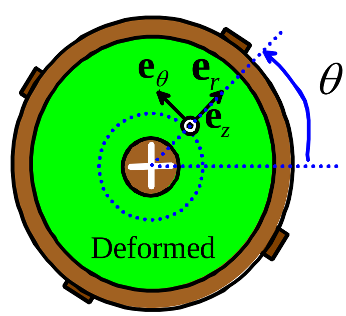
$$\sigma_{ij} = \frac{-3P_k y_k y_i y_j}{4\pi R^5} \qquad R = \sqrt{y_k y_k}$$

represents the stress in an infinite, incompressible linear elastic solid that is subjected to a point force with components P_k acting at the origin (you can visualize a point force as a very large body force which is concentrated in a very small region around the origin).

(a) Verify that the stress field is in static equilibrium

(b) Consider a spherical region of material centered at the origin. This region is subjected to (1) the body force acting at the origin; and (2) a force exerted by the stress field on the outer surface of the sphere. Calculate the resultant force exerted on the outer surface of the sphere by the stress, and show that it is equal in magnitude and opposite in direction to the body force.

3. The figure shows a test designed to measure the viscosity of a fluid. The sample is a hollow cylinder with internal radius a_0 and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated with angular velocity $\omega(t)$. Assume that all material particles in the specimen (green) move circumferentially, with a velocity field (in spatial coordinates) $\mathbf{v} = v_\theta(r,t)\mathbf{e}_\theta$.



(a) Calculate the spatial velocity gradient **L** in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and hence deduce the stretch rate tensor **D**.

(b) Calculate the acceleration field

(c) Suppose that the specimen is homogeneous, has mass density ρ, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

$$\boldsymbol{\tau} = 2\mu\mathbf{D} + p(r,t)\mathbf{I}$$

where p is a hydrostatic pressure (to be determined) and μ is the viscosity. Use this to write down an expression for the Cauchy stress tensor in terms of p, expressing your answer as components in $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

(d) Assume steady deformation. Express the equations of equilibrium in terms of $v_\theta(r,t)$.

(e) Solve the equilibrium equation, together with appropriate boundary conditions, to calculate $v_\theta(r,t)$, and *p(r)*. (The pressure can only be determined to within an arbitrary constant).

(f) Find an expression for the torque (per unit out of plane distance) necessary to rotate the external cylinder

(g) Calculate the rate of external work done by the torque acting on the rotating exterior cylinder

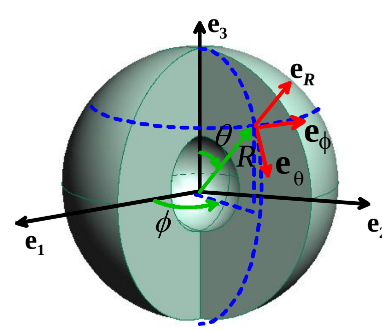
(h) Calculate the rate of internal dissipation in the solid as a function of *r*.

(i) Show that the total internal dissipation is equal to the rate of work done by the external moment.

4. A solid with volume V is subjected to a distribution of traction t_i on its surface. Assume that the solid is in static equilibrium (this requires that t_i exerts no resultant force or moment on the boundary). By considering a virtual velocity of the form $\delta v_i = A_{ij} y_j$, where A_{ij} is a constant tensor, use the principle of virtual work to show that the average stress in a solid can be computed from the shape of the solid and the tractions acting on its surface using the expression

$$\frac{1}{V}\int_V \sigma_{ij} dV = \frac{1}{V}\int_S \frac{1}{2}\left(t_i y_j + t_j y_i\right) dA$$

5. The shell shown in the figure is subjected to a radial body force $\mathbf{b} = \rho b(R)\mathbf{e}_R$, and a radial pressure p_a, p_b acting on the surfaces at $R = a$ and $R = b$. The loading induces a spherically symmetric state of stress in the shell, which can be expressed in terms of its components in a spherical-polar coordinate system as $\sigma_{RR}\mathbf{e}_R \otimes \mathbf{e}_R + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi$. By considering a virtual velocity of the form $\delta\mathbf{v} = w(R)\mathbf{e}_R$, show that the stress state is in static equilibrium if



$$\int_a^b \left\{\sigma_{RR}\frac{dw}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{w}{R}\right\} 4\pi R^2 dR - \int_a^b b(R) w(R) 4\pi R^2 dR - 4\pi a^2 p_a w(a) + 4\pi b^2 p_b w(b) = 0$$

for all *w(R)*. Hence, show that the stress state must satisfy

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b = 0 \qquad \sigma_{RR} = -p_a \;\; (R = a) \qquad \sigma_{RR} = -p_b \;\; (R = b)$$

6. An ideal gas with mass density ρ, pressure p and temperature θ has specific internal energy, specific Helmholtz free energy, and stress given by

$$\varepsilon = c_v \theta = \frac{p}{(\gamma - 1)\rho} \qquad \psi = c_v \theta - \theta\left(c_v \log\theta - R\log\rho - s_0\right) \qquad \sigma_{ij} = -p\delta_{ij} = -\rho R\theta\delta_{ij}$$

where R is the gas constant, c_v is the specific heat capacity (a positive constant), and s_0 is an arbitrary constant. Ideal gases are also characterized by the specific heat at constant pressure $c_p = c_v + R$ and the ratio $\gamma = c_p / c_v$. In addition, heat conduction through an ideal gas is often modeled using Fourier's law

$$q_i = -\kappa \frac{\partial \theta}{\partial y_i}$$

where κ is the thermal conductivity (a positive constant). Show that this constitutive model obeys the free energy imbalance

$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial \theta}{\partial y_i} - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \geq 0$$