School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 5: Application of continuum mechanics to fluids
Due 4pm Wednesday Nov 21

1. Starting with the local version of the first law of thermodynamics

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij} D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

and using the mass conservation equation

derive the statement of the first law of thermodynamics for a control volume

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int_R \rho\mathbf{b}\cdot\mathbf{v}dV - \int_B \mathbf{q}\cdot\mathbf{n}dA + \int_R q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n}dA$$

2. Idealize the air above the earth's surface as an ideal gas, with temperature distribution $\theta = \theta_0 - \lambda y_3$, where y_3 is the height above the earth's surface, θ_0 is the temperature at the earth's surface, and λ is a constant.

(a) Assuming the air is at rest, write down the simplified versions of the momentum balance equation and the constitutive equations for the air

(b) Compute the pressure and density distributions as a function of height above the surface, in terms of the pressure p_0 at the earth's surface. What happens in the limit $\lambda \to 0$?

3. Show that the energy equation for a compressible, inviscid fluid flow can be expressed in the form

$$\rho \left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial p}{\partial t}\right|_{\mathbf{x}} - \frac{\partial q_i}{\partial y_i} + q$$

where $h = \varepsilon + p/\rho$ is the specific enthalpy, and p is the pressure.

4. The incompressible Navier-Stokes equations are sometimes re-written in so-called 'Impetus-Gage' form. This is done by introducing an arbitrary scalar field ψ (called the 'gage' and then defining a vector field **m** (called the 'impetus') as

$$m_i = v_i - \frac{\partial \psi}{\partial y_i}$$

With these definitions, show that the governing equations for **m** (mass conservation and the incompressible Navier-Stokes equation) can be expressed as

$$\left.\frac{\partial m_i}{\partial t}\right|_{\mathbf{y}=const} + \in_{ijk} v_j \in_{kpq} \frac{\partial m_q}{\partial y_p} = -\frac{\partial}{\partial y_i}\left(\left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{y}=const} - \frac{\eta}{\rho}\frac{\partial^2 \psi}{\partial y_k \partial y_k} + \frac{p}{\rho} + \frac{1}{2} v_k v_k + \Phi \right) + \frac{\eta}{\rho}\frac{\partial^2 m_i}{\partial y_k \partial y_k}$$

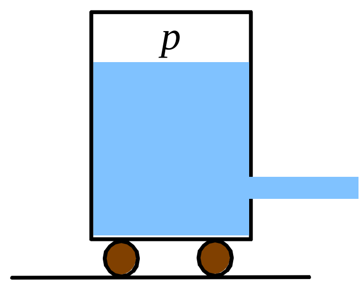
$$\frac{\partial m_i}{\partial y_i} = -\frac{\partial^2 \psi}{\partial y_i \partial y_i}$$

where $\Phi : b_i = -\partial \Phi / \partial y_i$ is the body force potential (The point of doing this is that since ψ is arbitrary, it can be chosen to satisfy any auxiliary equation that simplifies the governing equations for a particular example. For example, one could choose

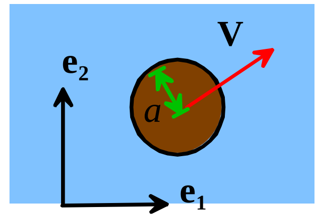
$$\left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{y}=const} - \frac{\eta}{\rho}\frac{\partial^2 \psi}{\partial y_k \partial y_k} + \frac{p}{\rho} + \frac{1}{2} v_k v_k + \Phi = 0$$

which reduces the governing equation for **m** to a very simple form - especially for ideal fluids)

5. The figure shows a pressurized soda can on a cart. The internal pressure above the fluid is p . A hole with cross-sectional area A is punched in the side of the can. Calculate the instantaneous acceleration of the cart, in terms of the pressure p, the surrounding atmospheric pressure p_a, and the combined mass m of the cart, can and fluid. Gravity can be neglected. You can assume that the cart is at rest if you wish, but the instantaneous acceleration is actually independent of the velocity.



6. The flow surrounding a rigid sphere (with radius a) that is at the origin at time $t=0$ and moves steadily without rotation with velocity V_i can be computed from the following potential



$$\phi = -\frac{a^3 V_i (y_i - V_i t)}{2r^3} \qquad r = \sqrt{(y_k - V_k t)(y_k - V_k t)}$$

a. Calculate the velocity field

b. Verify that the velocity field satisfies the correct boundary conditions on the surface of the sphere.

c. Calculate the pressure distribution (neglect gravity)

d. Hence, compute an expression for the distribution of traction acting on the surface of the sphere.

e. Determine the drag force acting on the sphere.

7. Consider a solid object (e.g. the sphere in the preceding problem) that moves through an ideal fluid with velocity $V_i(t)$ (not necessarily constant). The motion of the solid induces some velocity field v_i in the fluid, which can be calculated from a flow potential ϕ in the usual way. Show that the total kinetic energy of the fluid can be computed from

$$KE = -\frac{\rho}{2} \int_S \phi V_i(t) n_i dA$$

where ϕ is the flow potential, S is the surface of the solid object, and n_i is the outward normal to the solid surface. You will need to use the governing equation for the flow potential and the divergence theorem… You will also need to assume something about the behavior of the velocity field at infinity.

Hence, calculate the KE of the fluid surrounding a sphere moving with instantaneous velocity $V_i(t)$. Find an expression for the acceleration of a sphere with density ρ_s immersed in an ideal fluid (the buoyancy force can be treated without derivations….)