School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 5: Application of continuum mechanics to fluids
Due 12:00 noon Friday February 4th

1. Starting with the local version of the first law of thermodynamics

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij} D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

and using the mass conservation equation

derive the statement of the first law of thermodynamics for a control volume

$$\int_B (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot \mathbf{v} dA + \int_R \rho \mathbf{b}\cdot\mathbf{v} dV - \int_B \mathbf{q}\cdot\mathbf{n} dA + \int_R q dV = \frac{d}{dt}\int_R \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) dV + \int_B \rho\left(\varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\cdot\mathbf{n} dA$$

Start with the local version of the first law and integrate over a fixed spatial volume

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij} D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

Note that

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial \rho\varepsilon}{\partial t}\right|_{\mathbf{x}=const} - \varepsilon \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const}$$

recalling mass conservation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho \frac{\partial v_i}{\partial y_i} = 0$$

we see that

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial \rho\varepsilon}{\partial t}\right|_{\mathbf{x}=const} + \rho\varepsilon \frac{\partial v_i}{\partial y_i} = 0$$

Next

$$\left.\frac{\partial \rho\varepsilon}{\partial t}\right|_{\mathbf{x}=const} + \rho\varepsilon \frac{\partial v_i}{\partial y_i} = \left.\frac{\partial \rho\varepsilon}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho\varepsilon}{\partial y_i} v_i + \rho\varepsilon \frac{\partial v_i}{\partial y_i} = \left.\frac{\partial \rho\varepsilon}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho\varepsilon v_i}{\partial y_i}$$

Then

$$D_{ij}\sigma_{ij} = \sigma_{ij}\frac{\partial v_i}{\partial y_j} = \frac{\partial(\sigma_{ij} v_i)}{\partial y_j} - v_i \frac{\partial \sigma_{ij}}{\partial y_j}$$

Use the momentum balance equation

$$\frac{\partial \sigma_{ji}}{\partial y_j} = \rho\left(\frac{\partial v_i}{\partial y_k} v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const}\right) - \rho b_i$$

which gives

$$D_{ij}\sigma_{ij} = \sigma_{ij}\frac{\partial v_i}{\partial y_j} = \frac{\partial(\sigma_{ij} v_i)}{\partial y_j} - \rho v_i\left(\frac{\partial v_i}{\partial y_k} v_k + \left.\frac{\partial v_i}{\partial t}\right|_{y_i=const} - b_i\right)$$

Next, observe that

$$\frac{1}{2}\frac{\partial(\rho v_i v_i)}{\partial t}\bigg|_{\mathbf{y}=const} + \frac{1}{2}\frac{\partial \rho(v_i v_i v_k)}{\partial y_k} = \frac{v_i v_i}{2}\left(\frac{\partial \rho}{\partial t}\bigg|_{\mathbf{y}} + \frac{\partial(\rho v_k)}{\partial y_k}\right) + \rho v_i \frac{\partial v_i}{\partial t}\bigg|_{\mathbf{y}} + \rho v_k v_i \frac{\partial v_i}{\partial y_k}$$

$$= \rho v_i \frac{\partial v_i}{\partial t}\bigg|_{\mathbf{y}} + \rho v_k v_i \frac{\partial v_i}{\partial y_k}$$

from mass conservation. Substituting back, we have that

$$\frac{\partial \rho\varepsilon}{\partial t}\bigg|_{\mathbf{y}=const} + \frac{\partial \rho\varepsilon v_i}{\partial y_i} = \frac{\partial(\sigma_{ij} v_i)}{\partial y_j} - \frac{1}{2}\frac{\partial \rho v_k v_i v_i}{\partial y_k} - \frac{1}{2}\frac{\partial(\rho v_i v_i)}{\partial t}\bigg|_{\mathbf{y}=const} + v_i \rho b_i - \frac{\partial q_i}{\partial y_i} + q$$

Integrating this result over a fixed spatial volume, and then applying the divergence theorem yields the result stated.

[5 POINTS]

2. Idealize the air above the earth's surface as an ideal gas, with temperature distribution $\theta = \theta_0 - \lambda y_3$, where y_3 is the height above the earth's surface, θ_0 is the temperature at the earth's surface, and λ is a constant.

(a) Assuming the air is at rest, write down the simplified versions of the momentum balance equation and the constitutive equations for the air

Assuming the y_3 direction is vertical, the linear momentum balance equation reduces to

$$-\frac{\partial p}{\partial y_3} - \rho g = 0$$

The constitutive equations are

$$\varepsilon = c_v \theta = \frac{p}{(\gamma - 1)\rho} \qquad \psi = c_v \theta - \theta\left(c_v \log\theta - R\log\rho - s_0\right) \qquad p = \rho R \theta$$

[2 POINTS]

(b) Compute the pressure and density distributions as a function of height above the surface, in terms of the pressure p_0 at the earth's surface. What happens in the limit $\lambda \to 0$?

Substituting for the density, the pressure equation reduces to

$$-\frac{\partial p}{\partial y_3} - \frac{gp}{R(\theta_0 - \lambda y_3)} = 0 \;\Rightarrow\; \frac{p}{p_0} = \left(1 - \frac{\lambda y_3}{\theta_0}\right)^{g/R\lambda}$$

The density follows as

$$\rho = \frac{p_0}{R}\left(1 - \frac{\lambda y_3}{\theta_0}\right)^{g/(R\lambda)-1}$$

[2 POINTS]

The $\lambda = 0$ limit is not obvious by looking at this formula but it can easily be computed by solving the differential equation again, giving

$$p = p_0 \exp(-g y_3 / R\theta_0)$$

[1 POINT]

3. Show that the energy equation for a compressible, inviscid fluid flow can be expressed in the form

$$\rho \left.\frac{\partial h}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial p}{\partial t}\right|_{\mathbf{x}} - \frac{\partial q_i}{\partial y_i} + q$$

where $h = \varepsilon + p/\rho$ is the specific enthalpy, and p is the pressure.

The energy equation is

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \sigma_{ij} D_{ij} - \frac{\partial q_i}{\partial y_i} + q$$

The stress is $-p\delta_{ij}$ and so

$$\sigma_{ij} D_{ij} = -p \frac{\partial v_i}{\partial y_i} = \frac{p}{\rho}\frac{\partial \rho}{\partial t}$$

Where we have used mass conservation. Therefore

$$\rho \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} - \frac{p}{\rho}\frac{\partial \rho}{\partial t} = \rho \frac{\partial}{\partial t}(\varepsilon + \frac{p}{\rho}) - \frac{\partial p}{\partial t} = -\frac{\partial q_i}{\partial y_i} + q$$

$$\Rightarrow \rho \frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} - \frac{\partial q_i}{\partial y_i} + q$$

[3 POINTS]

4. The incompressible Navier-Stokes equations are sometimes re-written in so-called ‘Impetus-Gage’ form. This is done by introducing an arbitrary scalar field ψ (called the ‘gage’ and then defining a vector field **m** (called the ‘impetus’) as

$$m_i = v_i - \frac{\partial \psi}{\partial y_i}$$

With these definitions, show that the governing equations for **m** (mass conservation and the incompressible Navier-Stokes equation) can be expressed as

$$\left.\frac{\partial m_i}{\partial t}\right|_{\mathbf{y}=const} + \in_{ijk} v_j \in_{kpq} \frac{\partial m_q}{\partial y_p} = -\frac{\partial}{\partial y_i}\left(\left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{y}=const} - \frac{\eta}{\rho}\frac{\partial^2 \psi}{\partial y_k \partial y_k} + \frac{p}{\rho} + \frac{1}{2} v_k v_k + \Phi \right) + \frac{\eta}{\rho}\frac{\partial^2 m_i}{\partial y_k \partial y_k}$$

$$\frac{\partial m_i}{\partial y_i} = -\frac{\partial^2 \psi}{\partial y_i \partial y_i}$$

where $\Phi : b_i = -\partial \Phi / \partial y_i$ is the body force potential (The point of doing this is that since ψ is arbitrary, it can be chosen to satisfy any auxiliary equation that simplifies the governing equations for a particular example. For example, one could choose

$$\left.\frac{\partial \psi}{\partial t}\right|_{\mathbf{y}=const} - \frac{\eta}{\rho}\frac{\partial^2 \psi}{\partial y_k \partial y_k} + \frac{p}{\rho} + \frac{1}{2} v_k v_k + \Phi = 0$$

which reduces the governing equation for **m** to a very simple form – especially for ideal fluids)
The incompressible Navier-Stokes equation is

$$-\frac{1}{\rho}\frac{\partial p}{\partial y_i}+\frac{\eta}{\rho}\frac{\partial^2 v_i}{\partial y_j \partial v_j}+b_i=\left.\frac{dv_i}{dt}\right|_{\mathbf{x}=const}=\left.\frac{\partial v_i}{\partial t}\right|_{y_k=const}+\frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k)+\in_{ijk}\omega_j v_k$$

We have that

$$m_i+\frac{\partial \psi}{\partial y_i}=v_i$$

Substituting into the Navier-Stokes eq gives

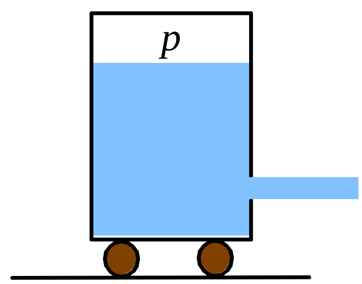
$$-\frac{1}{\rho}\frac{\partial p}{\partial y_i}+\frac{\eta}{\rho}\frac{\partial^2}{\partial y_j \partial y_j}\left(m_i+\frac{\partial \psi}{\partial y_i}\right)+b_i$$

$$=\left.\frac{\partial}{\partial t}\left(m_i+\frac{\partial \psi}{\partial y_i}\right)\right|_{y_k=const}+\frac{1}{2}\frac{\partial}{\partial y_i}(v_k v_k)+\in_{ijk}\in_{jpq}\frac{\partial}{\partial y_p}\left(m_q+\frac{\partial \psi}{\partial y_q}\right)v_k$$

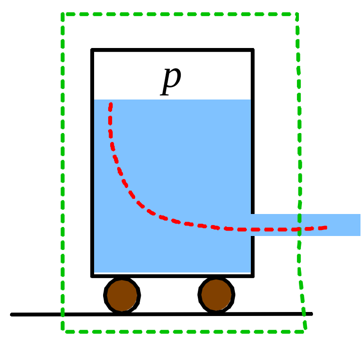
$$\Rightarrow \left.\frac{\partial m_i}{\partial t}\right|_{\mathbf{y}=const}=\in_{ikj} v_k \in_{jpq}\frac{\partial m_q}{\partial y_p}-\frac{\partial}{\partial y_i}\left(\frac{\partial \psi}{\partial y_i}-\frac{\eta}{\rho}\frac{\partial^2 \psi}{\partial y_j \partial y_j}+\frac{p}{\rho}+\frac{v_k v_k}{2}+\Phi\right)+\frac{\eta}{\rho}\frac{\partial^2 m_i}{\partial y_j \partial y_j}$$

[5 POINTS]

5. The figure shows a pressurized soda can on a cart. The internal pressure above the fluid is p . A hole with cross-sectional area A is punched in the side of the can. Calculate the instantaneous acceleration of the cart, in terms of the pressure p, the surrounding atmospheric pressure p_a, and the combined mass m of the cart, can and fluid. Gravity can be neglected. You can assume that the cart is at rest if you wish, but the instantaneous acceleration is actually independent of the velocity.



Consider the control volume shown in the figure. Linear momentum balance gives



$$\int_B \mathbf{n}\cdot\boldsymbol{\sigma}dA+\int_R \rho\mathbf{b}dV=\frac{d}{dt}\int_R \rho\mathbf{v}dV+\int_B(\rho\mathbf{v})\mathbf{v}\cdot\mathbf{n}dA$$

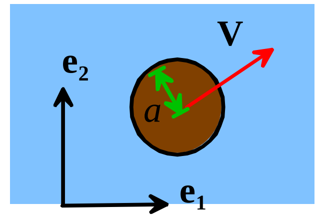
$$\Rightarrow 0=ma+A\rho v^2$$

Applying Bernoulli along the streamline shown in red gives

$p=p_a+\frac{1}{2}\rho v^2$. Therefore $a=2(p-p_a)A/m$

[3 POINTS]

6. The flow surrounding a rigid sphere (with radius *a*) that is at the origin at time *t=0* and moves steadily without rotation with velocity V_i can be computed from the following potential



$$\phi = -\frac{a^3 V_i (y_i - V_i t)}{2r^3} \qquad r = \sqrt{(y_k - V_k t)(y_k - V_k t)}$$

a. Calculate the velocity field

$$v_i = \frac{\phi}{\partial y_i} = -\frac{a^3 V_i}{2r^3} + 3\frac{a^3 V_k (y_k - V_k t)}{2r^4}\frac{(y_i - V_i t)}{r}$$

[2 POINTS]

b. Verify that the velocity field satisfies the correct boundary conditions on the surface of the sphere.

The boundary condition is

$$v_i n_i = V_i n_i$$

$$n_i = \frac{(y_i - V_i t)}{a}$$

$$v_i n_i = -\frac{a^3 V_i}{2r^3}\frac{(y_i - V_i t)}{r} + 3\frac{a^3 V_k (y_k - V_k t)}{2r^4}\frac{(y_i - V_i t)}{r}\frac{(y_i - V_i t)}{r} = V_i \frac{(y_i - V_i t)}{a}$$

[3 POINTS]

c. Calculate the pressure distribution (neglect gravity)

Using the Bernoulli equation:

$$\frac{p}{\rho} + \frac{1}{2}\frac{\partial \phi}{\partial y_i}\frac{\partial \phi}{\partial y_i} + \Phi + \frac{\partial \phi}{\partial t} = \frac{p_0}{\rho}$$

$$\frac{p}{\rho} + \frac{1}{2}\left(\frac{a^3 V_i}{2r^3}\frac{a^3 V_i}{2r^3} - 3\frac{a^3 V_k (y_k - V_k t)}{r^4}\frac{(y_i - V_i t)}{r}\frac{a^3 V_i}{2r^3} + 3\frac{a^3 V_k (y_k - V_k t)}{2r^4} 3\frac{a^3 V_n (y_n - V_n t)}{2r^4} \right)$$

$$+ \frac{a^3 V_i V_i}{2r^3} - 3\frac{a^3 V_k (y_k - V_k t)}{2r^4}\frac{(y_i - V_i t) V_i}{r} = \frac{p_0}{\rho}$$

$$\frac{p}{\rho} = \frac{p_0}{\rho} - \frac{a^3 V_i V_i}{2r^3} - \frac{a^6 V_i V_i}{8r^6} + 9\frac{a^3 V_k (y_k - V_k t)}{8r^4}\frac{(y_i - V_i t) V_i}{r}$$

[3 POINTS]

d. Hence, compute an expression for the distribution of traction acting on the surface of the sphere.

$$t_i = -p\delta_{ij} n_j = -\left[p_0 - \frac{5}{8}\rho V_k V_k + \frac{9}{8}\rho \left(\frac{V_k (y_k - V_k t)}{r} \right)^2 \right] \frac{(y_i - V_i t)}{r}$$

[2 POINTS]

e. Determine the drag force acting on the sphere.

The drag is zero by symmetry. This is not surprising – in an ideal fluid there is no dissipation. The solution is not very realistic, however, because it assumes that the flow does not separate from the surface of the sphere….

[2 POINTS]

7. Consider a solid object (e.g. the sphere in the preceding problem) that moves through an ideal fluid with velocity $V_i(t)$ (not necessarily constant). The motion of the solid induces some velocity field v_i in the fluid, which can be calculated from a flow potential ϕ in the usual way. Show that the total kinetic energy of the fluid can be computed from

$$KE = -\frac{\rho}{2}\int_S \phi V_i(t) n_i dA$$

where ϕ is the flow potential, S is the surface of the solid object, and n_i is the outward normal to the solid surface. You will need to use the governing equation for the flow potential and the divergence theorem… You will also need to assume something about the behavior of the velocity field at infinity.

The kinetic energy is

$$KE = \frac{\rho}{2}\int_V v_i v_i dV$$

Substituting for the velocity in terms of the flow potential

$$KE = \frac{\rho}{2}\int_V \frac{\partial \phi}{\partial y_i} v_i dV = \frac{\rho}{2}\int_V \frac{\partial \phi v_i}{\partial y_i} - \phi\frac{\partial v_i}{\partial y_i} dV = \frac{\rho}{2}\int_V \frac{\partial \phi v_i}{\partial y_i} dV$$

because the flow is incompressible. Applying the divergence theorem gives

$$KE = \frac{\rho}{2}\int_S \phi v_i m_i dA + \frac{\rho}{2}\int_{R_\infty} \phi v_i m_i dA$$

where R_∞ denotes some boundary very far away from the solid, and m_i is the outward normal to the *fluid* surface.. As long as the flow potential decays faster than 1/R at infinity, the contribution from the second integral must vanish. In addition, $m_i = -n_i$ This proves the statement.

Hence, calculate the KE of the fluid surrounding a sphere moving with instantaneous velocity $V_i(t)$. Find an expression for the acceleration of a sphere with density ρ_s immersed in an ideal fluid (the buoyancy force can be treated without derivations….)

[3 POINTS]

For the sphere,

$$KE = \frac{\rho}{2}\int_S \frac{a^3 V_i y_i}{2r^3}\frac{V_k y_k}{r} dA = \frac{\rho a^3 V_i V_k}{4}\int_S \frac{y_i y_k}{r^4} dA$$

This is the same integral that came up in HW3. It is nonzero only if $i = k$, and for this case the integral must be the same regardless of whether $i = k = 1, 2,$ or 3. We can just look up the solution… $4\pi\delta_{ij} / 3$
We thus get

$$KE = \frac{\rho\pi a^3 V_i V_i}{3} = \frac{W}{4g} V_i V_i$$

Where W is the weight of fluid displaced by the sphere.

[2 POINTS]

The buoyancy force on the sphere is $F_B = mg - W$ and the rate of work done by this force must equal the rate of change of potential energy of the system, which is

$$\left(\frac{W}{4g} + \frac{m}{2}\right)\frac{d|\mathbf{V}|^2}{dt} = \left(\frac{W}{2g} + m\right)\mathbf{a}\cdot\mathbf{V}$$

$$\Rightarrow a = \frac{2(mg - W)}{(W + 2mg)} g$$

[3 POINTS]