School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 6: Application of continuum mechanics to elastic solids

Due December 12th, 2012

1. Experiments show that rubber-like materials have specific internal energy $\varepsilon(\theta)$ and heat capacity $c(\theta)$ that are both essentially *independent* of strain **C**.

(a) Show that the heat capacity of an elastic material (regardless of the form of ε) is related to the specific entropy by

$$c = \theta \frac{\partial s}{\partial \theta}$$

$$\psi + \theta s = \varepsilon \Rightarrow c = \frac{\partial \varepsilon}{\partial \theta} = \frac{\partial \psi}{\partial \theta} + s + \theta \frac{\partial s}{\partial \theta}$$

Recall also that

$$s = -\frac{\partial \psi}{\partial \theta}$$

which then gives the answer stated.

(b) Hence, show that the entropy for rubber-like materials must have the separable form

$$s(\mathbf{C}, \theta) = g(\mathbf{C}) + h(\theta)$$

Integrating

$$c = \theta \frac{\partial s}{\partial \theta} \Rightarrow s = \int \frac{c}{\theta} d\theta + h(\mathbf{C})$$

(c) As a specific example, consider an incompressible, isotropic material of this kind, for which g is only a function of $\bar{I}_1 = \bar{B}_{kk}$ with $\bar{B}_{ij} = B_{ij} / J^{2/3}$ Show that Cauchy stress in such a material is given by

$$\sigma_{ij} = -2\rho_0 \theta \frac{dg}{dI_1}\left(B_{ij} - \frac{1}{3}\bar{I}_1 \delta_{ij} \right) + p\delta_{ij}$$

From notes we have that

$$\frac{\partial J}{\partial F_{ij}} = J F_{ji}^{-1} \qquad \frac{\partial \bar{I}_1}{\partial F_{ij}} = \frac{1}{J^{2/3}} \frac{\partial I_1}{\partial F_{ij}} - \frac{2 I_1}{3 J^{5/3}} \frac{\partial J}{\partial F_{ij}} = \frac{2}{J^{2/3}}\left(F_{ij} - \frac{I_1}{3} F_{ji}^{-1} \right) = \frac{2}{J^{2/3}} F_{ij} - \frac{2}{3} \bar{I}_1 F_{ji}^{-1}$$

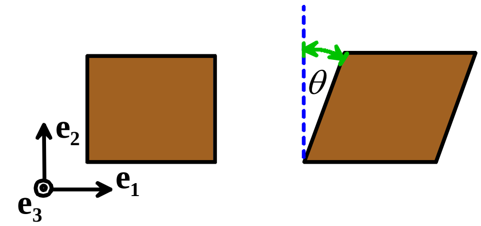
Furthermore $W = \rho_0 \psi = \rho_0 (\varepsilon - \theta s)$, and

$$\sigma_{ij} = \frac{1}{J} F_{ik} \frac{\partial W}{F_{jk}} = \frac{1}{J} F_{ik} \frac{\partial W}{\partial \bar{I}_1} \frac{\partial \bar{I}_1}{\partial F_{jk}} = -\rho_0 \theta \frac{\partial W}{\partial \bar{I}_1} F_{ik} \left(\frac{2}{J^{2/3}} F_{jk} - \frac{2}{3} \bar{I}_1 F_{kj}^{-1} \right)$$

Noting that *J=1* for an incompressible material gives the answer stated.

(d) Consider the simple shear deformation shown in the figure. Show that the shear stress in the solid is related to the shear strain $\gamma = \tan\theta$ by

$$\sigma_{12} = \mu\gamma \qquad \mu = -2\rho_0\theta \frac{dg}{dI_1}$$



(The generalized shear modulus must satisfy $\mu > 0$)

This follows by noting

$$[B] = \begin{bmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and substituting into the constitutive law

(e) Show that the uniaxial Cauchy stress-strain response of the material is given by

$$\sigma_{11} = \mu(\lambda^2 - 1/\lambda)$$

(the solid is loaded in uniaxial tension parallel to $\mathbf{e}_1$, and $\lambda = l/L$ is the principal stretch parallel to the loading axis, with l the deformed and L the underformed length of the bar, respectively)

For a volume preserving uniaxial deformation we have that

$$[B] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix}$$

The constitutive law therefore gives

$$[\sigma] = \mu\left(\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} - \frac{1}{3}\left(\lambda^2 + \frac{2}{\lambda}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) + p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the stress is uniaxial, it follows that

$$p = \frac{\mu}{3}\left(\lambda^2 + \frac{2}{\lambda}\right) - \frac{\mu}{\lambda}$$

and substituting back gives the result stated.

(f) Suppose that a bar of this rubber-like solid is loaded in uniaxial tension at a constant stress. How does the length of the bar change when its temperature is increased?

The modulus increases with temperature, so the bar gets *shorter* when heated.

(g) Suppose that the bar is stretched quasi-statically at constant temperature, with normalized extension rate $\dot{\lambda} = \frac{1}{L}\frac{dl}{dt}$ (neglect body forces). Show that the heat flow per unit volume into the bar is

$$Q = -\mu\left(\lambda^2 - \frac{1}{\lambda}\right)\frac{\dot{\lambda}}{\lambda}$$

(i.e. when stretched, the bar gives off heat)

The first law of thermodynamics gives $\frac{d}{dt}(\mathrm{E} + KE) = Q + W$, where

$$W = \int_V D_{ij}\sigma_{ij} dV$$

The stretch rate tensor is

$$[D] = [\dot{F}][F^{-1}] = \begin{bmatrix} \dot{\lambda} & 0 & 0 \\ 0 & -\dot{\lambda}/2\lambda^{3/2} & 0 \\ 0 & 0 & -\dot{\lambda}/2\lambda^{3/2} \end{bmatrix}\begin{bmatrix} 1/\lambda & 0 & 0 \\ 0 & \sqrt{\lambda} & 0 \\ 0 & 0 & \sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} \dot{\lambda}/\lambda & 0 & 0 \\ 0 & -\dot{\lambda}/2\lambda & 0 \\ 0 & 0 & -\dot{\lambda}/2\lambda \end{bmatrix}$$

Since the internal energy is a function only of temperature, and temperature is constant, it follows that

$$\frac{Q}{V} = -[D][\sigma]$$

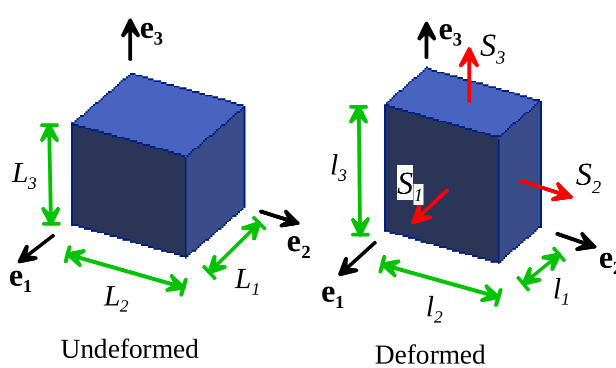
which gives the expression stated.

(h) Show that, if loaded as described in the preceding section under adiabatic conditions (no heat flow through the solid) the rate of change of temperature of the solid is

2. Derive the stress-strain relations for an incompressible, Neo-Hookean material subjected to

(a) Uniaxial tension
(b) Equibiaxial tension
(c) Pure shear

Derive expressions for the Cauchy stress, the Nominal stress, and the Material stress tensors (the solutions for nominal stress are listed in the notes). You should use the following procedure: (i) assume that the specimen experiences the length changes listed in the table in the notes ; (ii) use the stress-stretch relations to compute the Cauchy stress, leaving the hydrostatic part of the stress p as an unknown; (iii) Determine the hydrostatic stress from the boundary conditions (e.g. for uniaxial tensile parallel to $\mathbf{e}_1$ you know $\sigma_{22} = \sigma_{33} = 0$; for equibiaxial tension or pure shear in the $\mathbf{e}_1, \mathbf{e}_2$ plane you know that $\sigma_{33} = 0$)



Undeformed   Deformed

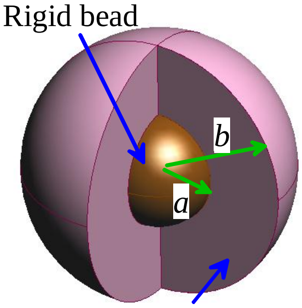
For the incompressible neo-hookean solid, the Cauchy stress-right stretch relation can be written as

$$\sigma_{ij} = \mu_1 B_{ij} + p\delta_{ij}$$

The deformation measures for the three cases are listed in the table below

	Uniaxial tension	Biaxial tension	Pure shear
	$l_1/L_1 = \lambda$ $l_2/L_2 = l_3/L_3 = \lambda^{-1/2}$ $I_1 = \lambda^2 + 2\lambda^{-1}$	$l_1/L_1 = l_2/L_2 = \lambda$ $l_3/L_3 = \lambda^{-2}$ $I_1 = 2\lambda^2 + \lambda^{-4}$	$l_1/L_1 = \lambda \quad l_2/L_2 = 1$ $l_3/L_3 = \lambda^{-1}$ $I_1 = 1 + \lambda^2 + \lambda^{-2}$
F	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}$	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$	$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$
B	$\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$	$\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^{-4} \end{bmatrix}$	$\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}$
p	$-\mu_1\lambda^{-1}$	$-\mu_1\lambda^{-4}$	$-\lambda^{-2}$
σ	$\mu_1\begin{bmatrix} \lambda^2 - \lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} \lambda^2 - \lambda^{-4} & 0 & 0 \\ 0 & \lambda^2 - \lambda^{-4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} \lambda^2 - \lambda^{-2} & 0 & 0 \\ 0 & 1 - \lambda^{-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
S	$\mu_1\begin{bmatrix} \lambda - \lambda^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} \lambda - \lambda^{-5} & 0 & 0 \\ 0 & \lambda - \lambda^{-5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} \lambda - \lambda^{-3} & 0 & 0 \\ 0 & \lambda^{-1} - \lambda^{-3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Σ	$\mu_1\begin{bmatrix} 1 - \lambda^{-3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} 1 - \lambda^{-6} & 0 & 0 \\ 0 & \lambda - \lambda^{-6} & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\mu_1\begin{bmatrix} 1 - \lambda^{-3} & 0 & 0 \\ 0 & \lambda^{-2} - \lambda^{-4} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

3. In a model experiment intended to duplicate the propulsion mechanism of the lysteria bacterium, a spherical bead with radius a is coated with an enzyme known as an "Arp2/3 activator." When suspended in a solution of actin, the enzyme causes the actin to polymerize at the surface of the bead. The polymerization reaction causes a spherical gel of a dense actin network to form around the bead. New gel is continuously formed at the bead/gel interface, forcing the rest of the gel to expand radially around the bead. The actin gel is a long-chain polymer and consequently can be idealized as a rubber-like incompressible neo-Hookean material. Experiments show that after reaching a critical radius the actin gel loses spherical symmetry and occasionally will fracture. Stresses in the actin network are believed to drive both processes. In this problem you will calculate the stress state in the growing, spherical, actin gel.



(a) Note that this is an unusual boundary value problem in solid mechanics, because a compatible reference configuration cannot be identified for the solid. Nevertheless, it is possible to write down a deformation gradient field that characterizes the change in shape of infinitesimal volume elements in the gel. To this end: (i) write down the length of a circumferential line at the surface of the bead; (ii) write down the length of a circumferential line at radius r in the gel; (iii) use these results, together with the incompressibility condition, to write down the deformation gradient characterizing the shape change of a material element that has been displaced from $r=a$ to a general position r. Assume that the bead is rigid, and that the deformation is spherically symmetric.

Length of a circumferential line at the surface of the bead is $L = 2\pi a$

Length of a circumferential line at radius r is $l = 2\pi r$

This gives $F_{\theta\theta} = F_{\phi\phi} = r/a$. For incompressibility, $F_{rr} = a^2 / r^2$

(b) Suppose that new actin polymer is generated at volumetric rate $\dot{V}$. Use the incompressibility condition to write down the velocity field in the actin gel in terms of $\dot{V}$, a and r (think about the volume of material crossing a radial line per unit time)

The volume of material inside a spherical shell with inner radius a and outer radius r must remain constant. This means that the volume of material flowing across a circumferential line at radius r must balance the volume being generated at radius a. Thus

$$4\pi r^2 v_r = \dot{V} \Rightarrow v_r = \dot{V} / (4\pi r^2)$$

(c) Calculate the velocity gradient $\mathbf{v} \otimes \nabla$ in the gel (i) by direct differentiation of (b) and (ii) by using the results of (a). Show that the results are consistent.

By direct differentiation, we have

$$[L]=\begin{bmatrix} \frac{\partial v_r}{\partial r} & 0 & 0 \\ 0 & \frac{v_r}{r} & 0 \\ 0 & 0 & \frac{v_r}{r} \end{bmatrix} = \begin{bmatrix} -\frac{2\dot{V}}{4\pi r^3} & 0 & 0 \\ 0 & \frac{\dot{V}}{4\pi r^3} & 0 \\ 0 & 0 & \frac{\dot{V}}{4\pi r^3} \end{bmatrix}$$

The alternative method is

$$[L]=[\dot{F}][F^{-1}]=\begin{bmatrix} \frac{-2a^2 v_r}{r^3} & 0 & 0 \\ 0 & \frac{v_r}{a} & 0 \\ 0 & 0 & \frac{v_r}{a} \end{bmatrix}\begin{bmatrix} \frac{r^2}{a^2} & 0 & 0 \\ 0 & \frac{a}{r} & 0 \\ 0 & 0 & \frac{a}{r} \end{bmatrix} = \begin{bmatrix} \frac{-2v_r}{r} & 0 & 0 \\ 0 & \frac{v_r}{r} & 0 \\ 0 & 0 & \frac{v_r}{r} \end{bmatrix}$$

(d) Calculate the components of the left Cauchy-Green deformation tensor field and hence write down an expression for the Cauchy stress field in the solid, in terms of an indeterminate hydrostatic pressure.

$$[B]=\begin{bmatrix} \frac{a^4}{r^4} & 0 & 0 \\ 0 & \frac{r^2}{a^2} & 0 \\ 0 & 0 & \frac{r^2}{a^2} \end{bmatrix}$$

The stress can be expressed as

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} B_{ij} + p\delta_{ij}$$

Where J=1. Thus

$$[\sigma]=\begin{bmatrix} \mu\frac{a^4}{r^4}+p & 0 & 0 \\ 0 & \mu\frac{r^2}{a^2}+p & 0 \\ 0 & 0 & \mu\frac{r^2}{a^2}+p \end{bmatrix}$$

(e) Use the equilibrium equations and boundary condition to calculate the full Cauchy stress distribution in the bead. Assume that the outer surface of the gel (at $r=b$) is traction free.

The equilibrium equation with spherical symmetry is

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + \rho_0 b_r = 0$$

This leads to

$$\frac{dp}{dr} + \frac{1}{r}\left(2\mu\frac{a^4}{r^4} - 2\mu\frac{r^2}{a^2}\right) = 0$$

Integrating this expression and using the boundary condition gives

$$p(r) = \mu(a^6b^4 - 3a^6r^4 - 2b^6r^4 + 3b^4r^6)/(2a^2b^4r^4)$$

The stresses then follow as

$$\sigma_{rr} = \mu(b^2 - r^2)(3a^6b^2 + 3a^6r^2 - 2b^4r^4)/(2a^2b^4r^4)$$

$$\sigma_{\theta\theta} = \sigma_{\phi\phi} = \mu(a^6b^4 - 3a^6r^4 - 2b^6r^4 + 4b^4r^6)/(2a^2b^4r^4)$$

4. A compressible, neo-Hookean solid has a stress-right C-G strain relation given by

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}}\left(B_{ij} - \frac{1}{3}B_{kk}\delta_{ij}\right) + K_1\left(J - 1\right)\delta_{ij}$$

Suppose that a solid consisting of such a material is first subjected to a deformation characterized by $F_{ij}^0, J_0 B_{ij}^0$, inducing a stress σ_{ij}^0. This deformation maps a material particle at position X_i in the reference configuration to position y_i in the deformed solid. The solid is then subjected to a further small deformation that induces an additional displacement distribution Δu_i in the material. Let

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial y_j} + \frac{\partial \Delta u_j}{\partial y_i}\right)$$

denote the increment of infinitesimal strain associated with this displacement, and expand the stress as a Taylor series in strain as

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl}\Delta\varepsilon_{kl} + \mathrm{O}(\Delta\varepsilon_{kl})^2$$

Show that the tangent modulus for this deformation is

(Hint: note, eg, that the Jacobian after the incremental deformation can be approximated as

$$J \approx J_0\left(1 + \frac{\partial \Delta u_m}{\partial y_m}\right)$$

The deformation gradient after the incremental displacement can be expressed as

$$F_{ij} = \left(\delta_{ip} + \frac{\partial \Delta u_i}{\partial y_p} \right) F_{pj}^0$$

$$\Rightarrow B_{ij} = F_{ik} F_{jk} = \left(\delta_{ip} + \frac{\partial \Delta u_i}{\partial y_p} \right) F_{pk}^0 \left(\delta_{jq} + \frac{\partial \Delta u_j}{\partial y_q} \right) F_{qk}^0$$

$$= \left(\delta_{ip} + \frac{\partial \Delta u_i}{\partial y_p} \right) B_{pq}^0 \left(\delta_{jq} + \frac{\partial \Delta u_j}{\partial y_q} \right) \approx B_{ij}^0 + B_{iq}^0 \frac{\partial \Delta u_j}{\partial y_q} + \frac{\partial \Delta u_i}{\partial y_p} B_{pj}^0$$

In addition

$$J \approx J_0 \left(1 + \frac{\partial \Delta u_m}{\partial y_m} \right) \qquad J^{-5/3} \approx J_0^{-5/3} \left(1 - \frac{5}{3} \frac{\partial \Delta u_m}{\partial y_m} \right)$$

Substituting these relations into the stress-strain relation, and noting that

$$\sigma_{ij}^0 = \frac{\mu_1}{J_0^{5/3}} \left(B_{ij}^0 - \frac{1}{3} B_{kk}^0 \delta_{ij} \right) + K_1 \left(J_0 - 1 \right) \delta_{ij}$$

shows that the stress can be expressed as

$$\sigma_{ij} = \sigma_{ij}^0 + \frac{\mu_1}{J_0^{5/3}} \left(B_{iq}^0 \frac{\partial \Delta u_j}{\partial y_q} + \frac{\partial \Delta u_i}{\partial y_p} B_{pj}^0 - \frac{1}{3} \left[B_{kq}^0 \frac{\partial \Delta u_k}{\partial y_q} + \frac{\partial \Delta u_k}{\partial y_p} B_{pk}^0 \right] \delta_{ij} \right)$$
$$- \frac{5}{3} \frac{\mu_1}{J_0^{5/3}} \left(B_{ij}^0 - \frac{1}{3} B_{nn}^0 \delta_{ij} \right) \frac{\partial \Delta u_k}{\partial y_k} + K_1 J_0 \frac{\partial \Delta u_k}{\partial y_k} \delta_{ij}$$

Noting that **B** is symmetric, this can be re-written as

$$\sigma_{ij} = \sigma_{ij}^0 + \left[\frac{\mu_1}{J_0^{5/3}} \left(B_{jl}^0 \delta_{ik} + B_{ik}^0 \delta_{jl} - \frac{2}{3} B_{kl}^0 \delta_{ij} \right) - \frac{5}{3} \frac{\mu_1}{J_0^{5/3}} \left(B_{ij}^0 - \frac{1}{3} B_{nn}^0 \delta_{ij} \right) \delta_{kl} + K_1 J_0 \delta_{kl} \delta_{ij} \right] \frac{\partial \Delta u_k}{\partial y_l}$$

5. A solid, spherical nuclear fuel pellet with outer radius a is subjected to a uniform internal distribution of heat due to a nuclear reaction. The heating induces a steady-state temperature field

$$T(r) = \left(T_a - T_0 \right) \frac{r^2}{a^2} + T_0$$

where T_0 and T_a are the temperatures at the center and outer surface of the pellet, respectively. Assume that the pellet can be idealized as a linear elastic solid with Young's modulus E, Poisson's ratio ν and thermal expansion coefficient α. Calculate the distribution of stress in the pellet.

The solution should follow the standard process.

(i) Calculate the displacement field by solving the following ODE

$$\frac{d^2 u}{dR^2} + \frac{2}{R} \frac{du}{dR} - \frac{2u}{R^2} = \frac{d}{dR} \left\{ \frac{1}{R^2} \frac{d}{dR} \left(R^2 u \right) \right\} = \frac{\alpha (1+\nu)}{(1-\nu)} \frac{d \Delta T}{dR} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \rho_0 b(R)$$

The solution contains two arbitrary constants – the first is determined by the condition that the displacement vanishes at the origin.

(ii) Determine the stress state from the constitutive law

$$\begin{bmatrix}\sigma_{RR}\\ \sigma_{\theta\theta}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & 2\nu\\ \nu & 1\end{bmatrix}\begin{bmatrix}\dfrac{du}{dR}\\ \dfrac{u}{R}\end{bmatrix}-\frac{E\alpha\Delta T}{1-2\nu}\begin{bmatrix}1\\1\end{bmatrix}$$

The second constant follows from the boundary condition $\sigma_{RR}=0$ at $r=a$. The algebra is somewhat tedious - A MATLAB solution (using mupad) is shown below.

```
assume(-1<`&nu;`<1/2)
assume(EE>0)
assume(a>0)
assume(r>0)
T:= DT*r^2/a^2:
ode1 := diff(diff(r^2*uu(r),r)/r^2,r)=`&alpha;

u := simplify(solve(ode({ode1,uu(0)=0}, uu(r))
srr := eval(EE/(1+`&nu;`)/(1-2*`&nu;`)*((1-`&r
C19val := solve(subs(srr,r=a),C19):
srrfull := simplify(subs(srr,C19=C19val))
```

$$\left\{\frac{\alpha\, DT\, EE\,(a^2-r^2)\,(\nu-3)}{5\,a^2\,(2\,\nu^2-3\,\nu+1)}\right\}$$

```
ufull := simplify(subs(u,C19=C19val))
```

$$\left\{-\frac{\alpha\, DT\, r\,(\nu\, r^2-4\,\nu\, a^2+2\,a^2+r^2)}{5\,a^2\,(\nu-1)}\right\}$$

```
sqq := EE/(1+`&nu;`)/(1-2*`&nu;`)*(`&nu;`*diff
sqqfull := simplify(subs(sqq,C19=C19val))
```

$$\left\{\frac{\alpha\, DT\, EE\,(\nu\, a^2+3\,\nu\, r^2-3\,a^2+r^2)}{5\,a^2\,(2\,\nu^2-3\,\nu+1)}\right\}$$