School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 1: Index Notation; basic tensor operations
Due September 21, 2016

1. Which of the following equations are valid expressions using index notation? If you decide an expression is invalid, state which rule is violated.

   (a) $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}$ (b) $\in_{ijk}\in_{kkj} = 0$ (c) $\rho \dfrac{\partial u_i}{\partial t} = \dfrac{\partial}{\partial x_j} C_{ijkl} \dfrac{\partial u_k}{\partial x_l}$

2. Let $R = \sqrt{x_k x_k}$. Calculate $\dfrac{\partial \log(R)}{\partial x_i}$ and $\dfrac{\partial^2 \log(R)}{\partial x_i \partial x_i}$.

3. Verify that $S_{ji}^{-1} = \dfrac{1}{2\det(\mathbf{S})} \in_{ipq}\in_{jkl} S_{pk}S_{ql}$ (i.e. use the index notation rules to show that $S_{ji}^{-1} S_{im} = \delta_{jm}$)

4. Use index notation rules to show that $\nabla \times \nabla \times \mathbf{u} = \nabla\nabla \cdot \mathbf{u} - \nabla^2 \mathbf{u}$

5. Let $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be a (not necessarily orthonormal) basis in $\mathbb{R}^3$, and let $g_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j$, $\mathbf{T} = T^{ij} \mathbf{m}_i \otimes \mathbf{m}_j$ $\mathbf{S} = S^{ij} \mathbf{m}_i \otimes \mathbf{m}_j$ $\mathbf{U} = \mathbf{ST}$. Find U^{ij} such that $\mathbf{U} = U^{ij} \mathbf{m}_i \otimes \mathbf{m}_j$, in terms of g_{ij}, T^{ij}, S^{ij} .

6. Let $\mathbf{a}, \mathbf{b}$ be two (not necessarily orthogonal) unit vectors in $\mathbb{R}^3$. Find formulas for the eigenvalues and eigenvectors of $\mathbf{S} = \mathbf{a} \otimes \mathbf{a} + \mathbf{b} \otimes \mathbf{b}$, in terms of $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. (Don't use the standard formulas to do this. You can write down one eigenvalue and eigenvector by inspection. This and the symmetry of **S** then tells you something about the direction of the other two eigenvectors. You can use that insight to find the remaining eigenvectors, and finally deduce the eigenvalues).

7. Let $\mathbf{R}$ be a proper orthogonal tensor (det(**R**)>0). Let I_1, I_2, I_3 be the three invariants of **R** defined in class. Show that $I_1 = I_2$ (but do this without any index notation manipulations).

8. Let **n** be the dual vector of a skew tensor **W** . What is $\mathbf{Wn}$?

9. Let $\mathbf{W}$ be a skew tensor. Show that $\mathbf{I}+\mathbf{W}$ is nonsingular and $(\mathbf{I}-\mathbf{W})(\mathbf{I}+\mathbf{W})^{-1}$ is orthogonal.

10. Let $\mathbf{S}$ be a non-singular second order tensor with invariants I_1, I_2, I_3 . Show that $\mathbf{S}^{-1} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I}) / I_3$

11. Let $\mathbf{S}$ be symmetric and $\mathbf{W}$ skew. Calculate $\mathbf{S} : \mathbf{W}$