School of Engineering
Brown University

# EN2210: Continuum Mechanics

## Homework 2: Polar and Curvilinear Coordinates, Kinematics
## Solutions

1. The for the vector $v_i = \dfrac{x_i}{R^3}$ and tensor $S_{ij} = \dfrac{\delta_{ij}}{R^3} + \dfrac{x_i x_j}{R^5}$ $\quad R = \sqrt{x_k x_k}$ , calculate:
   a. Their components in spherical-polar coordinates
   b. The gradient of **v** in spherical-polar coordinates
   c. The divergence of **S** in spherical-polar coordinates

(a) Note that $x_i / R = \mathbf{e}_R$ so we $\mathbf{v} = \dfrac{1}{R^2}\mathbf{e}_R$

For $S_{ij}$ the first term is isotropic so $\dfrac{\delta_{ij}}{R^3} + \dfrac{x_i x_j}{R^5} = \dfrac{1}{R^3}(\mathbf{e}_R \otimes \mathbf{e}_R + \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi) + \dfrac{1}{R^3}\mathbf{e}_R \otimes \mathbf{e}_R$

**[3 POINTS]**

(b) $$\nabla \mathbf{v} = \left( \frac{\partial}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial}{\partial \theta}\mathbf{e}_\theta + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}\mathbf{e}_\phi \right)\frac{1}{R^2}\mathbf{e}_R = -\frac{2}{R^3}\mathbf{e}_R \otimes \mathbf{e}_R + \frac{1}{R^3}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi)$$

**[2 POINTS]**

(c) It is easiest to use the formula

$$\nabla \cdot \mathbf{S} \equiv \begin{bmatrix} \dfrac{\partial S_{RR}}{\partial R} + 2\dfrac{S_{RR}}{R} + \dfrac{1}{R}\dfrac{\partial S_{\theta R}}{\partial \theta} + \cot\theta \dfrac{S_{\theta R}}{R} + \dfrac{1}{R\sin\theta}\dfrac{\partial S_{\phi R}}{\partial \phi} - \dfrac{1}{R}\left(S_{\theta\theta} + S_{\phi\phi}\right) \\ \dfrac{\partial S_{R\theta}}{\partial R} + 2\dfrac{S_{R\theta}}{R} + \dfrac{1}{R}\dfrac{\partial S_{\theta\theta}}{\partial \theta} + \cot\theta \dfrac{S_{\theta\theta}}{R} + \dfrac{1}{R\sin\theta}\dfrac{\partial S_{\phi\theta}}{\partial \phi} + \dfrac{S_{\theta R}}{R} - \cot\theta \dfrac{S_{\phi\phi}}{R} \\ \dfrac{\partial S_{R\phi}}{\partial R} + 2\dfrac{S_{R\phi}}{R} + \dfrac{\sin\theta}{R}\dfrac{\partial S_{\theta\phi}}{\partial \theta} + \cos\theta \dfrac{S_{\theta\phi}}{R} + \dfrac{1}{R\sin\theta}\dfrac{\partial S_{\phi\phi}}{\partial \phi} + \dfrac{1}{R}\left(S_{\phi R} + S_{\phi\theta}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial S_{RR}}{\partial R} + 2\dfrac{S_{RR}}{R} - \dfrac{1}{R}\left(S_{\theta\theta} + S_{\phi\phi}\right) \\ + \cot\theta \dfrac{S_{\theta\theta}}{R} - \cot\theta \dfrac{S_{\phi\phi}}{R} \\ 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{6}{R^4} + \dfrac{4}{R^4} - \dfrac{2}{R^4} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{4}{R^4} \\ 0 \\ 0 \end{bmatrix}$$

**[2 POINTS]**

2. 'Parabolic Coordinates' are used to simplify the solution of PDEs for solids with parabolic boundaries. They specify the position of a point using three parametric coordinates $(u,v,\theta)$ as

$$\mathbf{r} = uv(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \frac{1}{2}(u^2 - v^2)\mathbf{k}$$

(a) Find the components of normalized basis vectors for this coordinate system

$$\mathbf{e}_u = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial u}\right|}\frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{e}_v = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial v}\right|}\frac{\partial \mathbf{r}}{\partial u} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta}$$

Show that they are orthogonal; and calculate their derivatives with respect to $(u,v,\theta)$ (express your answer in $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_\theta\}$ coordinates.

$$\mathbf{e}_u = \frac{1}{\sqrt{v^2+u^2}}\left(v\cos\theta\mathbf{i} + v\sin\theta\mathbf{j} + u\mathbf{k}\right)$$

$$\mathbf{e}_v = \frac{1}{\sqrt{v^2+u^2}}\left(u\cos\theta\mathbf{i} + u\sin\theta\mathbf{j} - v\mathbf{k}\right)$$

$$\mathbf{e}_\theta = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

These are clearly orthogonal

Note that $\mathbf{k} = \frac{1}{\sqrt{u^2+v^2}}(u\mathbf{e}_u - v\mathbf{e}_v) \qquad \cos\theta\mathbf{i} + \sin\theta\mathbf{j} = \frac{1}{\sqrt{u^2+v^2}}(v\mathbf{e}_u + u\mathbf{e}_v)$

$$\frac{\partial \mathbf{e}_u}{\partial u} = -\frac{u}{(v^2+u^2)^{3/2}}\left(v\cos\theta\mathbf{i} + v\sin\theta\mathbf{j} + u\mathbf{k}\right) + \frac{1}{\sqrt{v^2+u^2}}\mathbf{k}$$

$$= -\frac{u}{(v^2+u^2)}\mathbf{e}_u + \frac{1}{(v^2+u^2)}(u\mathbf{e}_u - v\mathbf{e}_v) = -\frac{v}{v^2+u^2}\mathbf{e}_v$$

$$\frac{\partial \mathbf{e}_u}{\partial v} = -\frac{v}{(v^2+u^2)^{3/2}}\left(v\cos\theta\mathbf{i} + v\sin\theta\mathbf{j} + u\mathbf{k}\right) + \frac{1}{\sqrt{v^2+u^2}}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= -\frac{v}{(v^2+u^2)}\mathbf{e}_u + \frac{1}{v^2+u^2}(v\mathbf{e}_u + u\mathbf{e}_v) = \frac{u}{v^2+u^2}\mathbf{e}_v$$

$$\frac{\partial \mathbf{e}_u}{\partial \theta} = \frac{1}{\sqrt{v^2+u^2}}\left(-v\sin\theta\mathbf{i} + v\cos\theta\mathbf{j}\right) = \frac{v}{\sqrt{v^2+u^2}}\mathbf{e}_\theta$$

By symmetry

$$\frac{\partial \mathbf{e}_v}{\partial u} = \frac{v}{v^2+u^2}\mathbf{e}_u \qquad \frac{\partial \mathbf{e}_v}{\partial v} = -\frac{u}{v^2+u^2}\mathbf{e}_u \qquad \frac{\partial \mathbf{e}_v}{\partial \theta} = \frac{u}{\sqrt{v^2+u^2}}\mathbf{e}_\theta$$

And finally

$$\frac{\partial \mathbf{e}_\theta}{\partial u} = \frac{\partial \mathbf{e}_\theta}{\partial v} = 0 \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -\frac{1}{\sqrt{u^2+v^2}}(v\mathbf{e}_u + u\mathbf{e}_v)$$

**[5 POINTS]**

(b) Find an expression for the gradient operator in the $(u,v,\theta)$ system (express your answer in $\{\mathbf{e}_u,\mathbf{e}_v,\mathbf{e}_\theta\}$ coordinates.

Consider a scalar function $\mathrm{f}(u,v,\theta)$. Its derivative is $d\,\mathrm{f} = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial \theta}d\theta$

Similarly, note that by definition

$$d\mathbf{r} = \sqrt{u^2+v^2}\,\mathbf{e}_u du + \sqrt{u^2+v^2}\,\mathbf{e}_v dv + uv\mathbf{e}_\theta d\theta$$

Since $\{\mathbf{e}_u,\mathbf{e}_v,\mathbf{e}_\theta\}$ are orthogonal we find that

$$d\,\mathrm{f} = \left( \frac{1}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{uv}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta \right)\cdot d\mathbf{r}$$

The gradient operator therefore follows as

$$\nabla f = \left( \frac{1}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u}\mathbf{e}_u + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial}{\partial v}\mathbf{e}_v + \frac{1}{uv}\frac{\partial}{\partial \theta}\mathbf{e}_\theta \right) f$$

**[3 POINTS]**

(c) Find the gradient and divergence of the vector field $\mathbf{w} = \frac{1}{u^2}\mathbf{e}_u$

$$\nabla\mathbf{w} = \left( \frac{1}{\sqrt{u^2+v^2}}\frac{\partial}{\partial u}\mathbf{e}_u + \frac{1}{\sqrt{u^2+v^2}}\frac{\partial}{\partial v}\mathbf{e}_v + \frac{1}{uv}\frac{\partial}{\partial \theta}\mathbf{e}_\theta \right)\frac{1}{u^2}\mathbf{e}_u$$

$$= -\frac{2}{u^3\sqrt{u^2+v^2}}\mathbf{e}_u\otimes\mathbf{e}_u - \frac{1}{u^2\sqrt{u^2+v^2}}\frac{v}{v^2+u^2}\mathbf{e}_v\otimes\mathbf{e}_u$$

$$+\frac{1}{u^2\sqrt{u^2+v^2}}\frac{u}{v^2+u^2}\mathbf{e}_v\otimes\mathbf{e}_v + \frac{1}{u^3 v}\frac{v}{\sqrt{v^2+u^2}}\mathbf{e}_\theta\otimes\mathbf{e}_\theta$$

The divergence is the trace of this:

$$\nabla\cdot\mathbf{w} = \frac{1}{\sqrt{u^2+v^2}}\left( \frac{-2}{u^3} + \frac{1}{u^3} + \frac{1}{u(v^2+u^2)} \right)$$

$$= \frac{1}{\sqrt{u^2+v^2}}\left( -\frac{(v^2+u^2)}{u^3(v^2+u^2)} + \frac{u^2}{u^3(v^2+u^2)} \right) = \frac{v^2}{u^3(v^2+u^2)^{3/2}}$$

**[3 POINTS]**

3. Helical coordinates are used to reduce the governing equations for problems with helical symmetry (such as flow down a helical channel) to two dimensions. A number of different helical coordinate systems have been proposed. One example is defined by expressing position vector in terms of $(r,\theta,z) \equiv (\xi_1,\xi_2,\xi_3)$ as follows

$$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + (z + \frac{L}{2\pi}\theta)\mathbf{k}$$

Where $L$ is the pitch of the helix.

(a) Calculate the covariant basis vectors $\mathbf{m}_i$

$$\mathbf{m}_1 = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \qquad \mathbf{m}_2 = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} + \frac{L}{2\pi}\mathbf{k} \qquad \mathbf{m}_3 = \mathbf{k}$$

**[2 POINTS]**

(b) Find expressions for the reciprocal basis vectors $\mathbf{m}^i$

It is easiest to do this by inspection

$$\mathbf{m}^1 = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \qquad \mathbf{m}^2 = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})/r \qquad \mathbf{m}^3 = \frac{L}{2\pi r}(\sin\theta\mathbf{i} - \cos\theta\mathbf{j}) + \mathbf{k}$$

**[2 POINTS]**

(c) Calculate the covariant and contravariant components of the metric tensor $\mathbf{g}$ . Check your answer by calculating $g_{ik}g^{kj}$

$$g_{ij} = \mathbf{m}_i \cdot \mathbf{m}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 + (L/2\pi)^2 & L/2\pi \\ 0 & L/2\pi & 1 \end{bmatrix} \qquad g^{ij} = \mathbf{m}^i \cdot \mathbf{m}^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & -L/2\pi r^2 \\ 0 & -L/2\pi r^2 & 1 + (L/2\pi r)^2 \end{bmatrix}$$

It is easy to see that $g_{ik}g^{kj} = \delta_i^j$ as required.

**[2 POINTS]**

(d) Calculate the elements of Christoffel symbol (which satisfies $d\mathbf{m}_i = \Gamma_{ij}^k \mathbf{m}_k d\xi_j$ )

Note that $d\mathbf{m}_i \cdot \mathbf{m}^k = \Gamma_{ij}^l \mathbf{m}_l \cdot \mathbf{m}^k d\xi_j = \Gamma_{ij}^k d\xi_j$

Also

$d\mathbf{m}_1 = (-\sin\theta \mathbf{i} + \cos\theta \mathbf{j})\,\mathrm{d}\theta \qquad d\mathbf{m}_2 = -r(\cos\theta \mathbf{i} + \sin\theta \mathbf{j})\,\mathrm{d}\theta + (-\sin\theta \mathbf{i} + \cos\theta \mathbf{j})dr \qquad d\mathbf{m}_3 = \mathbf{0}$

$$d\mathbf{m}_1 \cdot \mathbf{m}^1 = 0 \qquad d\mathbf{m}_1 \cdot \mathbf{m}^2 = \frac{1}{r} d\theta \qquad d\mathbf{m}_1 \cdot \mathbf{m}^3 = -\frac{L}{2\pi r} d\theta$$
$$d\mathbf{m}_2 \cdot \mathbf{m}^1 = -r d\theta \qquad d\mathbf{m}_2 \cdot \mathbf{m}^2 = \frac{1}{r} dr \qquad d\mathbf{m}_2 \cdot \mathbf{m}^3 = -\frac{L}{2\pi r} dr$$
$$d\mathbf{m}_3 \cdot \mathbf{m}^1 = d\mathbf{m}_3 \cdot \mathbf{m}^2 = d\mathbf{m}_3 \cdot \mathbf{m}^3 = 0$$

The nonzero elements of $\Gamma_{ij}^k$ are thus

$$\Gamma_{12}^2 = \frac{1}{r} \qquad \Gamma_{12}^3 = -\frac{L}{2\pi r}$$
$$\Gamma_{22}^1 = -r \quad \Gamma_{21}^2 = \frac{1}{r} \quad \Gamma_{21}^3 = -\frac{L}{2\pi r}$$

**[3 POINTS]**

(e) Calculate expressions for the covariant and contravariant components of the gradient of a scalar function $\phi$

$$\nabla \phi = \mathbf{m}^1 \frac{\partial \phi}{\partial r} + \mathbf{m}^2 \frac{\partial \phi}{\partial \theta} + \mathbf{m}^3 \frac{\partial \phi}{\partial z}$$
$$\frac{\partial \phi}{\partial \xi_i} \mathbf{m}^i = g^{ij} \frac{\partial \phi}{\partial \xi_j} \mathbf{m}_j$$
$$\Rightarrow \nabla \phi = \mathbf{m}_1 \frac{\partial \phi}{\partial r} + \mathbf{m}_2 \left[ \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{L}{2\pi r^2} \frac{\partial \phi}{\partial z} \right] + \mathbf{m}_3 \left( -\frac{L}{2\pi r^2} \frac{\partial \phi}{\partial \theta} + (1 + \frac{L^2}{4\pi^2 r^2}) \frac{\partial \phi}{\partial z} \right)$$

**[2 POINTS]**

(f) Find expressions for the contravariant components of the velocity and acceleration of a particle, in terms of time derivatives of $r, \theta, z$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{dr}\frac{dr}{dt} + \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{dt} + \frac{d\mathbf{r}}{dz}\frac{dz}{dt} = \frac{dr}{dt}\mathbf{m}_1 + \frac{d\theta}{dt}\mathbf{m}_2 + \frac{dz}{dt}\mathbf{m}_3$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2 \xi_i}{dt^2}\mathbf{m}_i + \frac{d\xi_i}{dt}\frac{d\xi_j}{dt}\Gamma_{ij}^k \mathbf{m}_k$$
$$= \frac{d^2 r}{dt^2}\mathbf{m}_1 + \frac{d^2\theta}{dt^2}\mathbf{m}_2 + \frac{d^2 z}{dt^2}\mathbf{m}_3 - r\left(\frac{d\theta}{dt}\right)^2 \mathbf{m}_1 + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{m}_2 - 2\left(\frac{L}{2\pi r}\right)\frac{dr}{dt}\frac{d\theta}{dt}\mathbf{m}_3$$
$$= \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{m}_1 + \left[\frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{m}_2 + \left[\frac{d^2 z}{dt^2} - 2\left(\frac{L}{2\pi r}\right)\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{m}_3$$

**[2 POINTS]**

(g) A steady flow down a helical channel must have the form $\mathbf{v} = v(r)\mathbf{m}_2$ (note that $v$ does not have units of velocity because $\mathbf{m}_2$ is not a unit vector). Calculate the velocity gradient tensor

$$\nabla\mathbf{v} = \left( \frac{\partial v^i}{\partial \xi_k} \mathbf{m}_i + v^i \Gamma^j_{ik} \mathbf{m}_j \right) \otimes \mathbf{m}^k$$

$$\nabla\mathbf{v} = \frac{dv}{dr}\mathbf{m}_2 \otimes \mathbf{m}^1 + v\Gamma^1_{22}\mathbf{m}_1 \otimes \mathbf{m}^2 + v\Gamma^2_{21}\mathbf{m}_2 \otimes \mathbf{m}^1 + v\Gamma^3_{21}\mathbf{m}_3 \otimes \mathbf{m}^1$$

$$= \frac{dv}{dr}\mathbf{m}_2 \otimes \mathbf{m}^1 + v(-r)\mathbf{m}_1 \otimes \mathbf{m}^2 + \frac{v}{r}\mathbf{m}_2 \otimes \mathbf{m}^1 - \frac{vL}{2\pi r}\mathbf{m}_3 \otimes \mathbf{m}^1$$

**[2 POINTS]**

**4.** Construct (i.e. find a displacement field) a homogeneous deformation that has the following properties:

- The volume of the solid is doubled
- A material fiber parallel to the $\mathbf{e}_1$ direction in the undeformed solid increases its length by a factor of $\sqrt{2}$ and is oriented parallel to the $\mathbf{e}_1 + \mathbf{e}_2$ direction in the deformed solid
- A material fiber parallel to the $\mathbf{e}_2$ direction in the undeformed is oriented parallel to the $-\mathbf{e}_1 + \mathbf{e}_2$ direction in the deformed solid.
- A material fiber parallel to the $\mathbf{e}_3$ direction in the undeformed solid preserves its length and orientation in the deformed solid

There are several ways to do this problem. Here is one. We can express the deformation gradient as $\mathbf{F} = \mathbf{RU}$ . Without loss of generality we can assume the principal directions of **U** are parallel to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ , in which case

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{e}_i \otimes \mathbf{e}_i$$

The first, second and third conditions give $\lambda_1\lambda_2\lambda_3 = 2 \qquad \lambda_1 = \sqrt{2} \qquad \lambda_3 = 1$

Finally the first and second conditions show that **R** is a 45 degree counterclockwise rotation about the $\mathbf{e}_3$ axis. This gives

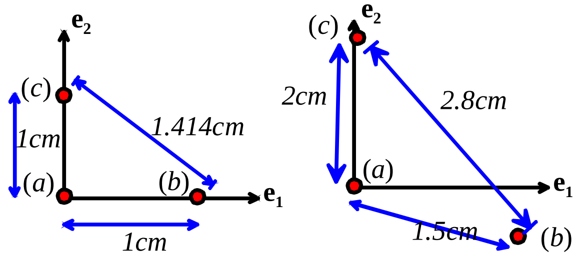
$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence **F** is

$$\mathbf{F} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The displacement field follows as $\mathbf{u} = \mathbf{Fx} + \mathbf{c}$ **[5 POINTS]**

**5. To** measure the in-plane deformation of a sheet of metal during a forming process, your managers place three small hardness indentations on the sheet. Using a travelling microscope, they determine that the initial lengths of the sides of the triangle formed by the three indents are 1cm, 1cm, 1.414cm, as shown in the picture below. After deformation, the sides have lengths 1.5cm, 2.0cm and 2.8cm.




5.1 Calculate the components of the Lagrange strain tensor $E_{11}$, $E_{22}$, $E_{12}$ in the basis shown.

By definition the Lagrange strain is related to the length changes by

$$\mathbf{m}\cdot\mathbf{E}\cdot\mathbf{m} = \frac{l^2 - l_0^2}{2l^2}$$

This gives

$$[1 \quad 0]\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{11} = \frac{1.5^2 - 1}{2} = 0.625$$

$$[0 \quad 1]\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = E_{22} = \frac{2^2 - 1}{2} = 1.5$$

$$\frac{1}{2}[1 \quad -1]\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (E_{11} + E_{22} - 2E_{12})/2 = \frac{2.8^2 - 2}{4} = 1.46 \Rightarrow E_{12} = -0.398$$

**[3 POINTS]**

5.2 Calculate the components of the Eulerian strain tensor $E_{11}^*$, $E_{22}^*$, $E_{12}^*$ in the basis shown.

There are various ways to do this. One way is to first calculate $\mathbf{F}$ , as follows

- Note that $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = 2\mathbf{E} + \mathbf{I}$
- This gives $\mathbf{U} = \begin{bmatrix} 1.482 & -0.2295 \\ -0.2295 & 1.9868 \end{bmatrix}$
- We know $\mathbf{F} = \mathbf{RU}$ and also that

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

- We see from the figure that the line $[0,1]$ is mapped to $[0,2]$ by deformation. We can solve the equation

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1.482 & -0.2295 \\ -0.2295 & 1.9868 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \theta = 0.115$$

- We can now calculate $\mathbf{F}$

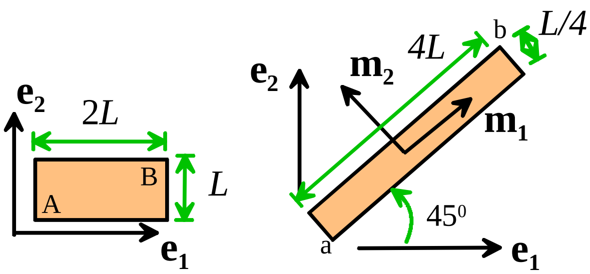
$$\mathbf{F} = \mathbf{RU} = \begin{bmatrix} 1.4464 & 0 \\ = 0.3975 & 2 \end{bmatrix}$$

- Finally

$$\mathbf{E}^* = (\mathbf{I} - \mathbf{F}^{-T}\mathbf{F}^{-1})/2$$
$$= \begin{bmatrix} 0.25155 & -0.03435 \\ -0.03435 & 0.375 \end{bmatrix}$$

**[5 POINTS]**

**6.** The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine




6.1 The right stretch tensor **U**, expressed as components in $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. (A 2x2 matrix is sufficient). There is no need for lengthy calculations – you may write down the result by inspection.

The deformed configuration can be reached by a stretch parallel to the two basis vectors, followed by a rotation. These can be taken to be the two deformations in the decomposition **F=RU**

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

**[2 POINTS]**

6.2 The rotation tensor **R** in the polar decomposition of the deformation gradient **F**=**RU**=**VR**

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

**[2 POINTS]**

6.3The deformation gradient, expressed as components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Try to do this without using the basis-change formulas.

The deformation gradient can be decomposed as **F=VR**, and **V** has components

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$, while **R** has the same components in both $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$. Therefore

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & -2 \\ 1/4 & 1/4 \end{bmatrix}$$

**[3 POINTS]**

**7.** Show that the Lagrange strain **E**, the right Cauchy-Green deformation tensor **C** and the right stretch tensor **U** have the same principal directions (eigenvectors). Similarly, show that $\mathbf{E}^*, \mathbf{B}, \mathbf{V}$ have the same principal directions.

We can write

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i$$

$$\Rightarrow \mathbf{C} = \mathbf{U}^2 = \sum_{i=1}^{3} \lambda_i^2 \mathbf{u}_i \otimes \mathbf{u}_i$$

$$\mathbf{E} = (\mathbf{C} - \mathbf{I}) / 2 = \sum_{i=1}^{3} (\lambda_i^2 - 1) \mathbf{u}_i \otimes \mathbf{u}_i / 2$$

Similarly

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}_i \otimes \mathbf{v}_i$$

$$\Rightarrow \mathbf{B} = \mathbf{V}^2 = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}_i \otimes \mathbf{v}_i$$

$$\mathbf{E}^* = (\mathbf{I} - \mathbf{B}^{-1}) / 2 = \sum_{i=1}^{3} (1 - 1 / \lambda_i^2) \mathbf{v}_i \otimes \mathbf{v}_i / 2$$

**[3 POINTS]**