School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 3: Kinematics
Solutions

1. The infinitesimal strain field in a long cylinder containing a hole at its center is given by

$$\varepsilon_{31} = -bx_2 / r^2 \qquad \varepsilon_{32} = bx_1 / r^2 \qquad r = \sqrt{x_1^2 + x_2^2}$$

(a) Show that the strain field satisfies the equations of compatibility.

The following compatibility conditions are not trivially satisfied:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} - \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = 0$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} \right) = 0$$

We can check these:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} - \frac{\partial}{\partial x_1}\left(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} \right) = b\frac{\partial}{\partial x_1}\left(-\frac{\partial x_1 / r^2}{\partial x_1} - \frac{\partial x_2 / r^2}{\partial x_2} \right) = b\frac{\partial}{\partial x_1}\left(\frac{1}{r^2} - 2\frac{x_1^2}{r^4} + \frac{1}{r^2} - 2\frac{x_2^2}{r^4} \right) = 0$$

$$\frac{\partial^2 \varepsilon_{22}}{\partial x_3 \partial x_1} - \frac{\partial}{\partial x_2}\left(-\frac{\partial \varepsilon_{31}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3} + \frac{\partial \varepsilon_{23}}{\partial x_1} \right) = b\frac{\partial}{\partial x_2}\left(\frac{\partial x_2 / r^2}{\partial x_2} + \frac{\partial x_1 / r^2}{\partial x_1} \right) = b\frac{\partial}{\partial x_2}\left(\frac{\partial x_2 / r^2}{\partial x_2} + \frac{\partial x_1 / r^2}{\partial x_1} \right) = 0$$

[2 POINTS]

(b) Show that the strain field is consistent with a displacement field of the form $u_3 = \theta$, where $\theta = 2b\tan^{-1} x_2 / x_1$. Note that although the strain field is compatible, the displacement field is *multiple valued* – i.e. the displacements are not equal at $\theta = 2\pi$ and $\theta = 0$, which supposedly represent the same point in the solid. Of course, displacement fields like this do exist in solids – they are caused by dislocations in a crystal.

$$\varepsilon_{31} = \frac{1}{2}\frac{\partial u_3}{\partial x_1} = b\frac{\partial}{\partial x_1}\tan^{-1}\frac{x_2}{x_1} = -b\frac{x_2}{r^2}$$

$$\varepsilon_{32} = \frac{1}{2}\frac{\partial u_3}{\partial x_2} = b\frac{\partial}{\partial x_2}\tan^{-1}\frac{x_2}{x_1} = b\frac{x_1}{r^2}$$

[2 POINTS]

2. Calculate the displacement field that generates the following 3D infinitesimal strain field

$$\varepsilon_{ij} = (1+\nu)(x_k x_k \delta_{ij} + 2x_i x_j) - (3-\nu)\delta_{ij}$$

(it is easier to do this using the method of integrating strain components than the formal path integral)

We have that

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = (1+\nu)\left(3x_1^2 + x_2^2 + x_3^2\right) - (3-\nu)$$

$$\Rightarrow u_1 = (1+\nu)(x_1^3 + (x_2^2 + x_3^2)x_1) - (3-\nu)x_1 + f_1(x_2, x_3)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = 3(1+\nu)\left(3x_2^2 + x_1^2 + x_3^2\right) - (3-\nu)$$

$$\Rightarrow u_2 = (1+\nu)(x_2^3 + (x_1^2 + x_3^2)x_2) - (3-\nu)x_2 + f_2(x_1, x_3)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} = 3(1+\nu)\left(3x_3^2 + x_1^2 + x_2^2\right) - (3-\nu)$$

$$\Rightarrow u_3 = (1+\nu)(x_3^3 + (x_1^2 + x_2^2)x_3) - (3-\nu)x_3 + f_3(x_1, x_2)$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + 2(1+\nu)x_2 x_1 = 2(1+\nu)x_2 x_1 \Rightarrow \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{\partial f_1}{\partial x_3} + \frac{\partial f_3}{\partial x_1} + 2(1+\nu)x_3 x_1 = 2(1+\nu)x_3 x_1 \Rightarrow \frac{\partial f_1}{\partial x_3} + \frac{\partial f_3}{\partial x_1} = 0$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} + 2(1+\nu)x_3 x_2 = 2(1+\nu)x_3 x_2 \Rightarrow \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0$$

The three shear strain equations yield

$$\frac{\partial f_1}{\partial x_2} = \omega_3 \qquad \frac{\partial f_2}{\partial x_1} = -\omega_3$$

$$\Rightarrow f_1 = a_1 + \omega_3 x_2 + g(x_3) \qquad f_2 = a_2 - \omega_3 x_1 + h(x_3)$$

$$\frac{\partial f_1}{\partial x_3} = \frac{\partial g}{\partial x_3} = -\omega_2 \qquad \frac{\partial f_3}{\partial x_1} = \omega_2 \Rightarrow$$

$$f_1 = a_1 + \omega_3 x_2 - \omega_2 x_3 \qquad f_3 = a_3 + \omega_2 x_1 + q(x_2)$$

$$\frac{\partial f_2}{\partial x_3} = \frac{\partial h}{\partial x_3} = \omega_1 \qquad \frac{\partial f_3}{\partial x_2} = \frac{\partial q}{\partial x_2} = -\omega_1$$

$$\Rightarrow f_2 = a_2 - \omega_3 x_1 + \omega_1 x_3 \qquad f_3 = a_3 + \omega_2 x_1 - \omega_1 x_2$$

(this assumes a-priori that the unknown functions describe a rigid rotation)

Simplifying gives

$$u_i = (1+\nu)(x_k x_k)x_i - (3-\nu)x_i + a_i + \in_{ijk} \omega_j x_k$$

[5 POINTS]

Alternatively if you don't want to assume the unknown functions of integration are a rigid rotation:

$$\frac{\partial f_1}{\partial x_2} = p_3(x_3) \qquad \frac{\partial f_2}{\partial x_1} = -p_3(x_3)$$

$$\frac{\partial f_1}{\partial x_3} = p_2(x_2) \qquad \frac{\partial f_3}{\partial x_1} = -p_2(x_2)$$

$$\frac{\partial f_2}{\partial x_3} = p_1(x_1) \qquad \frac{\partial f_3}{\partial x_2} = -p_1(x_1)$$

$$\Rightarrow f_1 = p_3 x_2 + q_3(x_3) \qquad f_1 = p_2 x_3 + q_2(x_2)$$

$$f_2 = -p_3 x_1 + g_3(x_3) \qquad f_2 = p_1 x_3 + g_1(x_1)$$

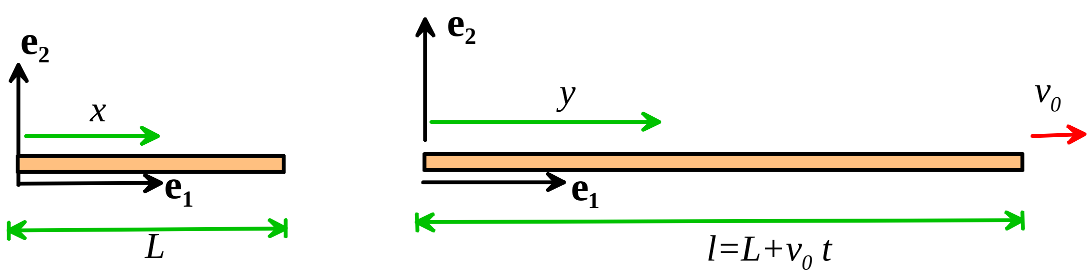
$$f_3 = -p_2 x_1 + h_2(x_2) \qquad f_3 = -p_1 x_2 + h_3(x_1)$$

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = \frac{\partial p_2}{\partial x_2} x_3 + \frac{\partial q_2}{\partial x_2} = -p_3 \Rightarrow \frac{\partial p_2}{\partial x_2} = 0, \ \frac{\partial q_2}{\partial x_2} = -p_3 = \text{constant}$$

$$\frac{\partial f_1}{\partial x_3} + \frac{\partial f_3}{\partial x_1} = \frac{\partial q_3}{\partial x_3} = -p_2 \Rightarrow \frac{\partial q_3}{\partial x_3} = -p_2 = \text{constant}$$

Continuing this argument yields the rigid motion as before….

3. A rubber band has initial length L. One end of the band is held fixed. For time $t>0$ the other end is pulled at constant speed v_0. Following the usual convention, let x denote position in the reference configuration, and let y denote position in the deformed configuration. Assume one dimensional deformation.



3.1 Write down the position y of a material particle as a function of its initial position x and time t.

$$y = \frac{L + v_0 t}{L} x$$

[1 POINT]

3.2 Hence, determine the velocity distribution as both a function of x and a function of y.

$$v = \frac{dl}{dt}\frac{x}{L} = \frac{v_0 x}{L} = \frac{v_0 y}{L + v_0 t}$$

[2 POINTs]

3.3 Find the deformation gradient (you only need to state the one nonzero component)

$$F = \frac{L + v_0 t}{L}$$

[1 POINT]

3.4 Find the velocity gradient

$$\frac{dv}{dy} = \frac{v_0}{L + v_0 t}$$

[1 POINT]

3.5 Suppose that a fly walks along the rubber band with speed *w* relative to the band. Calculate the acceleration of the fly as a function of time and other relevant variables.

$$a = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial y}\left(v + w\right) = -\frac{v_0^2 y}{(L + v_0 t)^2} + \frac{v_0}{L + v_0 t}\left(\frac{v_0 y}{L + v_0 t} + w\right) = \frac{v_0 w}{L + v_0 t}$$

[2 POINTS]

3.6 Suppose that the fly is at *x=y=0* at time *t=0*. Find how long it takes for the fly to walk to the other end of the rubber band, in terms of *L*, v_0 and *w*. It is easiest to do this by calculating dx/dt for the fly.

$$y = F_{11}x \Rightarrow \frac{dy}{dt} = \dot{F}_{11}x + F_{11}\dot{x}$$

$$\Rightarrow \frac{dx}{dt} = F_{11}^{-1}\left(\frac{dy}{dt} - \dot{F}_{11}x\right) = \frac{L}{L + v_0 t}\left(\frac{v_0}{L}x + w - \frac{v_0}{L}x\right) = \frac{w}{1 + v_0 t / L}$$

$$\Rightarrow x = \frac{wL}{v_0}\log(1 + v_0 t / L)$$

$$\Rightarrow t(x = L) = \frac{L}{v_0}\left(\exp(v_0 / w) - 1\right)$$

[4 POINTS]

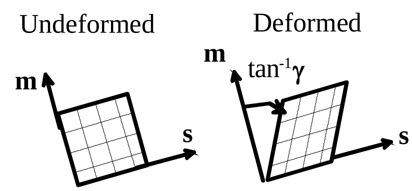
Also as a check, we see that

$$y = F_{11}x = \left(1 + \frac{v_0 t}{L}\right)\frac{wL}{v_0}\log(1 + v_0 t / L)$$

$$\Rightarrow \frac{dy}{dt} = w\log(1 + v_0 t / L) + w$$

$$\Rightarrow \frac{d^2 y}{dt^2} = \frac{v_0 w}{L + v_0 t}$$

4. A single crystal deforms by shearing on a single active slip system as illustrated in the figure. The crystal is loaded so that the slip direction $\mathbf{s}$ and normal to the slip plane $\mathbf{m}$ maintain a constant direction during the deformation



(a) Show that the deformation gradient can be expressed in terms of the components of the slip direction $\mathbf{s}$ and the normal to the slip plane $\mathbf{m}$ as $F_{ij} = \delta_{ij} + \gamma s_i m_j$ where γ denotes the shear, as illustrated in the figure.

(b) Suppose shearing proceeds at some rate $\dot{\gamma}$. At the instant when $\gamma = 0$, calculate (i) the velocity gradient tensor; (ii) the stretch rate tensor and (iii) the spin tensor associated with the deformation.

(c) Find an expression for the stretch rate and angular velocity of a material fiber parallel to a unit vector $\mathbf{n}$ in the deformed solid, in terms of $\dot{\gamma}, \mathbf{s}, \mathbf{m}$.

(a) A material fiber $d\mathbf{x}$ in the undeformed crystal becomes $d\mathbf{y} = d\mathbf{x} + \gamma \mathbf{s} d\mathbf{x} \cdot \mathbf{m} = \left(\mathbf{I} + \gamma \mathbf{s} \otimes \mathbf{m}\right) \cdot d\mathbf{x}$

This gives the deformation gradient.

[1 POINT]

(b)

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\gamma}\mathbf{s} \otimes \mathbf{m}$$

The velocity gradient is $\mathbf{D} = sym(\mathbf{L}) = \dot{\gamma}\frac{1}{2}\left(\mathbf{s} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{s}\right)$

$$\mathbf{W} = skew(\mathbf{L}) = \dot{\gamma}\frac{1}{2}\left(\mathbf{s} \otimes \mathbf{m} - \mathbf{m} \otimes \mathbf{s}\right)$$

[3 POINTS]

(c) We can write

$$\frac{d}{dt} l\mathbf{n} = \frac{dl}{dt}\mathbf{n} + l\frac{d\mathbf{n}}{dt} = \dot{\gamma}\mathbf{s}(\mathbf{m} \cdot \mathbf{n})l$$

$$\Rightarrow \frac{dl}{dt} = \dot{\gamma}(\mathbf{s} \cdot \mathbf{n})(\mathbf{m} \cdot \mathbf{n})l$$

$$\Rightarrow l\frac{d\mathbf{n}}{dt} = \dot{\gamma}\mathbf{s}(\mathbf{m} \cdot \mathbf{n})l - \dot{\gamma}(\mathbf{s} \cdot \mathbf{n})(\mathbf{m} \cdot \mathbf{n})l\mathbf{n} = \dot{\gamma}l(\mathbf{m} \cdot \mathbf{n})\left(\mathbf{s} - (\mathbf{s} \cdot \mathbf{n})\mathbf{n}\right)$$

We know that

$$\frac{d\mathbf{n}}{dt} = \boldsymbol{\omega} \times \mathbf{n} \Rightarrow \mathbf{n} \times \boldsymbol{\omega} \times \mathbf{n} = \boldsymbol{\omega} - (\mathbf{n} \cdot \boldsymbol{\omega})\mathbf{n} = \mathbf{n} \times \frac{d\mathbf{n}}{dt}$$

$$\Rightarrow \boldsymbol{\omega} = \dot{\gamma}(\mathbf{m} \cdot \mathbf{n})\left(\mathbf{n} \times \mathbf{s}\right)$$

where we have noted that the angular velocity must be orthogonal to **n**.

[2 POINTS]

5. Derive the identities relating accelerations to velocity gradient, stretch rate and vorticity

$$a_i = \left.\frac{\partial v_i}{dt}\right|_{\mathbf{y}} + \frac{1}{2}\frac{d}{dy_i}(v_k v_k) + 2W_{ij}v_j$$

$$\in_{ijk} \frac{\partial a_k}{\partial y_j} = \left.\frac{\partial \omega_i}{\partial t}\right|_{\mathbf{x}=const} - D_{ij}\omega_j + \frac{\partial v_k}{\partial y_k}\omega_i$$

For the first identity notethat

$$\left.\frac{\partial v_i}{dt}\right|_{\mathbf{y}} + \frac{1}{2}\frac{d}{dy_i}(v_k v_k) + 2W_{ij}v_j = \left.\frac{\partial v_i}{dt}\right|_{\mathbf{y}} + v_k\frac{dv_k}{dy_i} + 2\frac{1}{2}\left(\frac{dv_i}{dy_j} - \frac{dv_j}{dy_i}\right)v_j$$

$$= \left.\frac{\partial v_i}{dt}\right|_{\mathbf{y}} + \frac{dv_i}{dy_j}v_j = a_i$$

[2 POINTS]

For the second, start with the first identity and recall that the vorticity vectpr is half the dual vector of **W** so

$$a_i = \left.\frac{\partial v_i}{dt}\right|_{\mathbf{y}} + \frac{1}{2}\frac{d}{dy_i}(v_k v_k) + \in_{ijk} \omega_j v_k$$

We can take the curl of this expression (recall that the curl of a gradient is zero)

$$\in_{ijk} \frac{\partial a_k}{\partial y_j} = \in_{ijk} \frac{\partial}{\partial y_j}\left.\frac{\partial v_k}{dt}\right|_{\mathbf{y}} + \in_{ijk} \frac{\partial}{\partial y_j} \in_{klm} \omega_l v_m$$

$$= \in_{ijk} \frac{\partial}{\partial y_j}\left.\frac{\partial v_k}{dt}\right|_{\mathbf{y}} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial y_j}\omega_l v_m$$

$$= \in_{ijk} \frac{\partial}{\partial y_j}\left.\frac{\partial v_k}{dt}\right|_{\mathbf{y}} + \frac{\partial}{\partial y_j}(\omega_i v_j) - \frac{\partial}{\partial y_j}(\omega_j v_i)$$

$$= \left.\frac{\partial \omega_i}{dt}\right|_{\mathbf{y}} + \frac{\partial \omega_i}{\partial y_j}v_j + \omega_i\frac{\partial v_j}{\partial y_j} - v_i\frac{\partial \omega_j}{\partial y_j} - \frac{\partial v_i}{\partial y_j}\omega_j$$

$$= \left.\frac{\partial \omega_i}{dt}\right|_{\mathbf{x}} + \omega_i\frac{\partial v_j}{\partial y_j} - v_i\frac{\partial \omega_j}{\partial y_j} - \frac{\partial v_i}{\partial y_j}\omega_j$$

Recall that the divergence of a curl is zero, and note also $\mathbf{L\omega} = (\mathbf{D}+\mathbf{W})\mathbf{\omega} = \mathbf{D\omega}$, which gives the required answer.

[5 POINTS]

6. Show that $\mathbf{D} = \mathbf{F}^{-T} \dfrac{d\mathbf{E}}{dt}\mathbf{F}^{-1}$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}\mathbf{F}^T - \mathbf{I}) \Rightarrow \frac{d\mathbf{E}}{dt} = \frac{1}{2}\left(\frac{d\mathbf{F}^T}{dt}\mathbf{F} + \mathbf{F}^T \frac{d\mathbf{F}}{dt} \right)$$

$$\mathbf{D} = sym(\frac{d\mathbf{F}}{dt}\mathbf{F}^{-1}) = \frac{1}{2}\left(\mathbf{F}^{-T}\frac{d\mathbf{F}^T}{dt} + \frac{d\mathbf{F}}{dt}\mathbf{F}^{-1} \right)$$

[2 POINTS]

7. Let **n** be a unit vector parallel to infinitesimal material fiber in a deforming solid. Show that

$$\frac{d\mathbf{n}}{dt} = \mathbf{Dn} + \mathbf{Wn} - (\mathbf{n}\cdot\mathbf{Dn})\mathbf{n}$$

$$\begin{aligned}
\mathbf{n} &= \frac{d\mathbf{y}}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}} \Rightarrow \frac{d\mathbf{n}}{dt} = \frac{1}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}}\frac{d\mathbf{y}}{dt} - \frac{d\mathbf{y}}{(d\mathbf{y}\cdot d\mathbf{y})^{3/2}}\frac{d\mathbf{y}}{dt}\cdot d\mathbf{y} \\
&= (\mathbf{D}+\mathbf{W})\frac{d\mathbf{y}}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}} - \frac{d\mathbf{y}}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}}\frac{d\mathbf{y}}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}}\cdot\left(\mathbf{D}+\mathbf{W}\right)\frac{d\mathbf{y}}{\sqrt{d\mathbf{y}\cdot d\mathbf{y}}} \\
&= (\mathbf{D}+\mathbf{W})\mathbf{n} + (\mathbf{n}\cdot\mathbf{Dn})\mathbf{n}
\end{aligned}$$

(since $\mathbf{n}\cdot\mathbf{Wn} = 0$ because **W** is skew)

[2 POINTS]

8. Derive the transport formula

$$\frac{d}{dt}\int_S \phi n_i dA = \int_S \left(\delta_{ij}\left.\frac{\partial\phi}{\partial t}\right|_{\mathbf{x}=const} + \delta_{ij}\phi\frac{\partial v_k}{\partial y_k} - \phi\frac{\partial v_j}{\partial y_i} \right) n_j dA$$

Use the procedure described in the notes:

$$\frac{d}{dt}\int_S \phi n_i dA = \frac{d}{dt}\int_{S_0} \phi J F_{ji}^{-1} n_j^0 dA_0$$

$$\mathbf{F}\mathbf{F}^{-1} = \mathbf{I} \Rightarrow \dot{\mathbf{F}}\mathbf{F}^{-1} + \mathbf{F}\dot{\mathbf{F}}^{-1} = 0 \Rightarrow \dot{\mathbf{F}}^{-1} = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}$$

$$\dot{\mathbf{F}}^{-T} = -\mathbf{F}^{-T}\dot{\mathbf{F}}^{T}\mathbf{F}^{-T}$$

$$\int_{S_0} \frac{d}{dt}\left(\phi J F_{ji}^{-1}\right) n_j^0 dA_0 = \int_{S_0} \frac{d\phi}{dt}\left(JF_{ji}^{-1}\right) n_j^0 dA_0 + \int_{S_0} \phi J \nabla_y \cdot \mathbf{v}\left(F_{ji}^{-1}\right) n_j^0 dA_0 + \int_{S_0} \phi J\left(\dot{F}_{ji}^{-1}\right) n_j^0 dA_0$$

$$= \int_S \frac{d\phi}{dt} n_j dA + \int_S \phi\nabla_y \cdot \mathbf{v} n_j dA - \int_S \phi L_{ji} n_j dA$$

[5 POINTS]