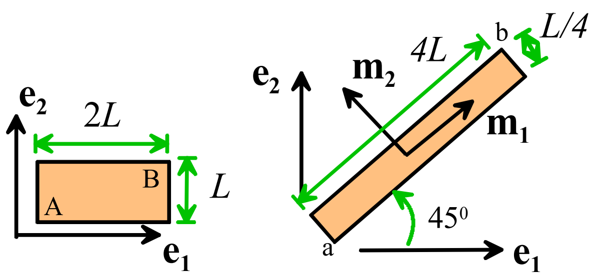
School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 4: Kinetics and Conservation Laws
Due Wednesday Oct 19, 2016

1. The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine



The Cauchy stress in the solid is $\sigma \mathbf{m}_1 \otimes \mathbf{m}_1$. Determine:

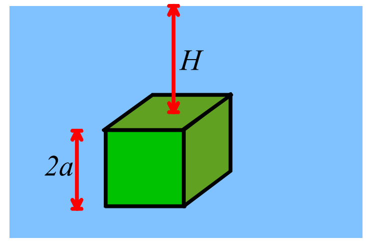
1.1 The components of Cauchy stress in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

1.2 The components of Nominal stress $\mathbf{S}$ in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. What is the nominal stress in the mixed basis $S_{ij}\mathbf{e}_i \otimes \mathbf{m}_j$?

1.3 The components of material stress in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

2. Show that the von-Mises effective stress $\sigma_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{DEV} : \boldsymbol{\sigma}^{DEV}}$ is invariant under a change of basis.

3. A rigid, cubic solid is immersed in a fluid with mass density ρ. Recall that a stationary fluid exerts a compressive pressure of magnitude $\rho g h$ at depth h.



3.1 Write down expressions for the traction vector exerted by the fluid on each face of the cube. You might find it convenient to take the origin for your coordinate system at the center of the cube, and take basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ perpendicular to the cube faces.

3.2 Calculate the resultant force due to the tractions acting on the cube, and show that the vertical force is equal and opposite to the weight of fluid displaced by the cube.

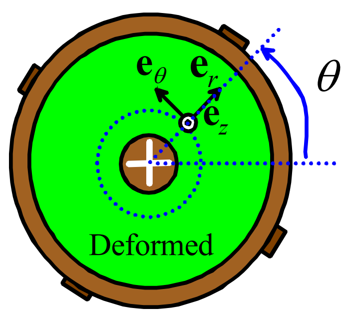
3.3 Show that the result of problem 3.3 applies to any arbitrarily shaped solid immersed below the surface of a fluid, i.e. prove that the resultant force acting on an immersed solid with volume V is $P_i = \rho g V \delta_{i3}$, where it is assumed that $\mathbf{e}_3$ is vertical.

4. Derive the result used in reducing the global equation of angular momentum conservation to local form

$$\frac{d}{dt}\int_V \mathbf{y} \times \rho \mathbf{v} dV = \int_V \mathbf{y} \times \rho \frac{d\mathbf{v}}{dt} dV$$

5. The figure shows an idealization of a shear viscometer, in which a fluid is confined between two coaxial rigid cylinders. The inner cylinder is fixed; the outer rotates with a constant angular rate. The stress distribution in the fluid is known to have the form $\boldsymbol{\sigma} = \sigma_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$, where $\sigma_{r\theta}$ is a function of r. Assume that the stress state is in static equilibrium and neglect body forces. By considering a virtual velocity field of the form $\mathbf{v} = \delta v_\theta(r)\mathbf{e}_\theta$, use the principle of virtual work to show that the equilibrium equation for the stress field reduces to

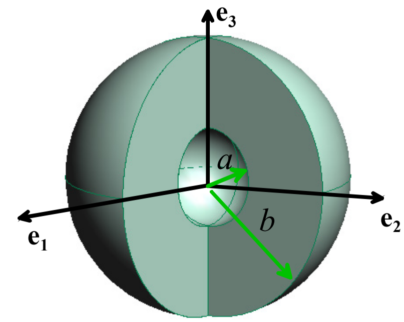
$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$



6. A thick walled spherical shell is made from an incompressible linear viscous material, in which the Cauchy stress is related to the stretch rate **D** by

$$\boldsymbol{\sigma} = 2\mu\mathbf{D} + p\mathbf{I}$$

where p is a hydrostatic stress to be determined and μ is a material property (viscosity).



The solid is subjected to a radial gravitational body force

$$\rho\mathbf{b} = -\rho B_0 \frac{r-a}{b-a}\mathbf{e}_r$$

6.1 Assume that the velocity field in the shell is radial $\mathbf{v} = v(r)\mathbf{e}_r$. Calculate the velocity gradient and stretch rate.

6.2 Show that the incompressibility condition implies that

$$\frac{\partial v}{\partial r} + \frac{2}{r}v = 0$$

and hence find an expression for $v(r)$ in terms of $v(a) = \dot{a}$.

6.3 Hence, find an expression for the particle acceleration $\left.\frac{dv}{dt}\right|_{\mathbf{x}}$, in terms of r, $\dot{a}$ and $\ddot{a}$ Be careful with this – it is not just the partial time derivative of $v(r)$.

6.4 Find an expression for the (total) rate of work done on the shell by gravity.

6.6 Find an expression for the total kinetic energy of the shell, in terms of $\dot{a}$

6.7 Calculate the time derivative of kinetic energy. Note that b is not constant.

6.8 Find the total internal stress power, in terms of $\dot{a}$, μ

6.9 Use the principle of virtual work to show that the stress state must satisfy

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) - \rho B_0 \frac{r-b}{a-b} = \rho \left.\frac{dv}{dt}\right|_{\mathbf{x}}$$

6.10 Write down the boundary conditions for $p(r)$ at $r = a,b$.

6.11 Hence, show that *a(t)* satisfies the differential equation

$$\frac{d^2a}{dt^2} = -B_0 \frac{b}{2a} - \frac{da}{dt}\frac{4\mu}{\rho}\frac{a^2+b^2+ab}{a^2b^2} - \left(\frac{da}{dt}\right)^2 \frac{(b-a)(a^2+2ab+3b^2)}{2ab^3}$$

6.12 Verify the result of 6.11 using energy methods

6.13 Plot $a(t)$ with initial conditions $a=1; b=5; \dot{a}=0$ for $B_0=1$ and $\mu/\rho = 0.1, 0.2, 0.4, 1.0, 2.0$ (you will need to solve the differential equation numerically, eg using Matlab or Mathematica).