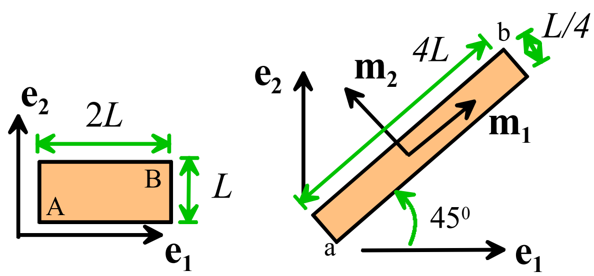
EN2210: Continuum Mechanics

Homework 4: Kinetics and Conservation Laws
Solutions

School of Engineering
Brown University

1. The figure shows the reference and deformed configurations for a solid. The out-of-plane dimensions are unchanged. Points a and b are the positions of points A and B after deformation. Determine



The Cauchy stress in the solid is $\sigma \mathbf{m}_1 \otimes \mathbf{m}_1$. Determine:

1.1 The components of Cauchy stress in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

We can write $\sigma \mathbf{m}_1 \otimes \mathbf{m}_1 = \frac{1}{2}\sigma(\mathbf{e}_1 + \mathbf{e}_2) \otimes (\mathbf{e}_1 + \mathbf{e}_2) = \frac{1}{2}\sigma \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{2}\sigma\left(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1\right) + \frac{1}{2}\sigma \mathbf{e}_2 \otimes \mathbf{e}_2$

[2 POINTS]

1.2 The components of Nominal stress **S** in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$. What is the nominal stress in the mixed basis $S_{ij}\mathbf{e}_i \otimes \mathbf{m}_j$?

Recall that by definition $\mathbf{S} = J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$

We can use the results from HW2:

- recall $\mathbf{F} = \mathbf{RU} = \mathbf{VR} \Rightarrow \mathbf{F}^{-1} = \mathbf{U}^{-1}\mathbf{R}^T = \mathbf{R}^T\mathbf{V}^{-1}$
- recall **V** has components in $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/4 \end{bmatrix}$$

- **R** has components

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

Therefore

$$\mathbf{R}^T\mathbf{V}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 4\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1/2 & 4\\ -1/2 & 4\end{bmatrix}$$

S follows as

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma} = \frac{1}{2}\frac{1}{\sqrt{2}}\begin{bmatrix}1/2 & 4\\ -1/2 & 4\end{bmatrix}\begin{bmatrix}\sigma & 0\\ 0 & 0\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}\sigma/2 & 0\\ -\sigma/2 & 0\end{bmatrix}$$

We can write this as

$$\begin{aligned}\mathbf{S} &= \frac{\sigma}{4\sqrt{2}}(\mathbf{m}_1 \otimes \mathbf{m}_1 - \mathbf{m}_2 \otimes \mathbf{m}_1)\\ &= \frac{\sigma}{4\sqrt{2}}\left[\frac{1}{2}(\mathbf{e}_1+\mathbf{e}_2)\otimes(\mathbf{e}_1+\mathbf{e}_2) - \frac{1}{2}(-\mathbf{e}_1+\mathbf{e}_2)\otimes(\mathbf{e}_1+\mathbf{e}_2)\right]\\ &= \frac{\sigma}{4\sqrt{2}}\frac{1}{2}\left[\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1 + \mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1\right]\\ &= \frac{\sigma}{4\sqrt{2}}\left[\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_1\right]\end{aligned}$$

In the mixed basis $\mathbf{S} = \frac{\sigma}{4}\mathbf{e}_1 \otimes \mathbf{m}_1$

[6 POINTS]

1.3 The components of material stress in both $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$

By definition $\boldsymbol{\Sigma} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} = J\mathbf{R}^T\mathbf{V}^{-1}\boldsymbol{\sigma}\mathbf{V}^{-1}\mathbf{R}$

Note that in the $\{\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ basis

$$J\mathbf{V}^{-1}\sigma\mathbf{V}^{-1} = \frac{1}{2}\begin{bmatrix}1/2 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}\sigma & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/2 & 0\\ 0 & 4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\sigma/4 & 0\\ 0 & 0\end{bmatrix} \equiv \frac{\sigma}{8}\mathbf{m}_1 \otimes \mathbf{m}_1$$

Note also $\mathbf{e}_i = \mathbf{R}^T\mathbf{m}_i$. Hence

$$\boldsymbol{\Sigma} = \frac{\sigma}{8}\mathbf{e}_1 \otimes \mathbf{e}_1$$

It follows that

$$\boldsymbol{\Sigma} = \frac{\sigma}{8}\frac{1}{2}(\mathbf{m}_1 - \mathbf{m}_2)\otimes(\mathbf{m}_1 - \mathbf{m}_2) = \frac{\sigma}{16}(\mathbf{m}_1\otimes\mathbf{m}_1 + \mathbf{m}_1\otimes\mathbf{m}_1 - \mathbf{m}_1\otimes\mathbf{m}_2 - \mathbf{m}_2\otimes\mathbf{m}_1)$$

[4 POINTS]

2. Show that the von-Mises effective stress $\sigma_e = \sqrt{\frac{3}{2}\boldsymbol{\sigma}^{DEV} : \boldsymbol{\sigma}^{DEV}}$ is invariant under a change of basis.

We have

$$\boldsymbol{\sigma}^{DEV} = \boldsymbol{\sigma} - trace(\boldsymbol{\sigma})\mathbf{I}/3$$

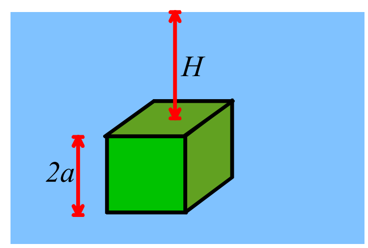
$$\boldsymbol{\sigma}^{DEV} : \boldsymbol{\sigma}^{DEV} = \boldsymbol{\sigma} : \boldsymbol{\sigma} - 2trace(\boldsymbol{\sigma})^2/3 + trace(\boldsymbol{\sigma})^2/3 = \boldsymbol{\sigma} : \boldsymbol{\sigma} - trace(\boldsymbol{\sigma})^2/3$$

In index notation

$$\sigma_e^{(\mathbf{m})} = \sqrt{Q_{ki}\sigma_{kl}Q_{lj}Q_{pi}\sigma_{pq}Q_{qj} - Q_{ki}\sigma_{kl}Q_{li}} = \sigma_e^{(\mathbf{e})}$$

[2 POINTS]

3. A rigid, cubic solid is immersed in a fluid with mass density ρ. Recall that a stationary fluid exerts a compressive pressure of magnitude $\rho g h$ at depth *h*.



3.1 Write down expressions for the traction vector exerted by the fluid on each face of the cube. You might find it convenient to take the origin for your coordinate system at the center of the cube, and take basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ perpendicular to the cube faces.

Let $\mathbf{e}_3$ be vertical, and origin at the center of the cube.

- Top face: $\mathbf{t} = -\rho g H \mathbf{e}_3$
- Bottom face $\mathbf{t} = \rho g(H + 2a)\mathbf{e}_3$
- Side faces $\mathbf{t} = -\rho g(H + a - x_3)\mathbf{e}_1,\ \rho g(H + a - x_3)\mathbf{e}_1, -\rho g(H + a - x_3)\mathbf{e}_2,\ \rho g(H + a - x_3)\mathbf{e}_2$

[2 POINTS]

3.2 Calculate the resultant force due to the tractions acting on the cube, and show that the vertical force is equal and opposite to the weight of fluid displaced by the cube.

The integrals over the side faces are zero (the tractions on opposite faces are equal and opposite).

The resultant force of pressure on the top and bottom faces is

$$\mathbf{F} = \left(-4\rho g H a^2 + 4\rho g(H + 2a)a^2\right)\mathbf{e}_3 = 8\rho g a^3 e_3$$

[2 POINTS]

3.3 Show that the result of problem 3.3 applies to any arbitrarily shaped solid immersed below the surface of a fluid, i.e. prove that the resultant force acting on an immersed solid with volume V is $P_i = \rho g V \delta_{i3}$, where it is assumed that $\mathbf{e}_3$ is vertical.

Let y_3 denote the coordinate perpendicular to the water surface. The stress state is $\sigma_{ij} = -\rho g y_3 \delta_{ij}$. The resultant force is

$$P_i = -\int_A \sigma_{ji} n_j dA = \int_A \rho g y_3 n_i dA = \int_V \rho g \frac{\partial}{\partial y_i} y_3 dV = \rho g V \delta_{i3}$$

[2 POINTS]

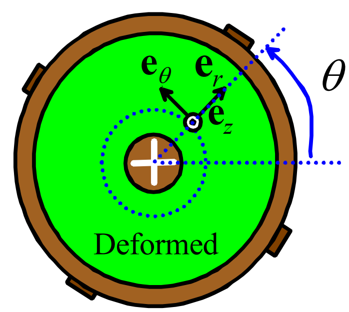
4. Derive the result used in reducing the global equation of angular momentum conservation to local form

$$\frac{d}{dt}\int_V \mathbf{y} \times \rho \mathbf{v} dV = \int_V \mathbf{y} \times \rho \frac{d\mathbf{v}}{dt} dV$$

$$\frac{d}{dt}\int_V \mathbf{y} \times \rho \mathbf{v} dV = \frac{d}{dt}\int_{V_0} \mathbf{y} \times \rho_0 \mathbf{v} dV_0 = \int_{V_0} \frac{d}{dt}(\mathbf{y} \times \rho_0 \mathbf{v}) dV_0 = \int_{V_0} \mathbf{y} \times \rho_0 \frac{d\mathbf{v}}{dt} dV_0 = \int_V \mathbf{y} \times \rho \frac{d\mathbf{v}}{dt} dV$$

[2 POINTS]

5. The figure shows an idealization of a shear viscometer, in which a fluid is confined between two coaxial rigid cylinders. The inner cylinder is fixed; the outer rotates with a constant angular rate. The stress distribution in the fluid is known to have the form $\boldsymbol{\sigma} = \sigma_{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r)$, where $\sigma_{r\theta}$ is a function of r. Assume that the stress state is in static equilibrium and neglect body forces. By considering a virtual velocity field of the form $\mathbf{v} = \delta v_\theta(r)\mathbf{e}_\theta$, use the principle of virtual work to show that the equilibrium equation for the stress field reduces to



$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$

The virtual work principle is

$$\int_V \boldsymbol{\sigma} : \delta \mathbf{D} = \int_S \mathbf{t}^* \cdot \delta \mathbf{v}$$

The velocity gradient is

$$\mathbf{L} = \begin{bmatrix} 0 & -\dfrac{\delta v_\theta}{r} \\ \dfrac{\partial \delta v_\theta}{\partial r} & 0 \end{bmatrix}$$

and so

$$\boldsymbol{\sigma} : \delta \mathbf{D} = \boldsymbol{\sigma} : \delta \mathbf{L} = \sigma_{r\theta}\left(\frac{\partial \delta v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

The virtual work equation therefore reduces to

$$\int_a^b \sigma_{r\theta}\left(\frac{\partial \delta v_\theta}{\partial r} - \frac{v_\theta}{r}\right) 2\pi r dr = 2\pi \delta v_\theta(b)\sigma_{r\theta}(b) - 2\pi a \delta v_\theta(a)\sigma_{r\theta}(a)$$

Integrate the first term by parts

$$\int_a^b r\sigma_{r\theta}\frac{\partial \delta v_\theta}{\partial r} = b\delta v_\theta(b)\sigma_{r\theta}(b) - a\delta v_\theta(a)\sigma_{r\theta}(a) - \int_a^b \frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right)\delta v_\theta dr$$

Hence

$$-\int_a^b \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right) + \frac{\sigma_{r\theta}}{r}\right)\delta v_\theta r dr = 0 \qquad \forall \delta v_\theta$$

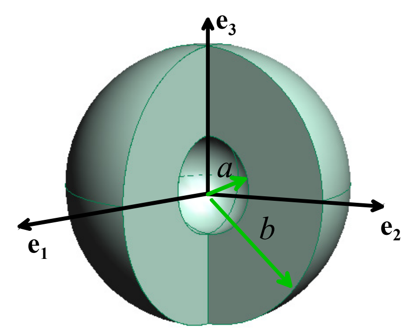
$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{r\theta}\right) + \frac{\sigma_{r\theta}}{r} = \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0$$

[5 POINTS]

6. A thick walled spherical shell is made from an incompressible linear viscous material, in which the Cauchy stress is related to the stretch rate **D** by

$$\boldsymbol{\sigma} = 2\mu \mathbf{D} + p\mathbf{I}$$

where p is a hydrostatic stress to be determined and μ is a material property (viscosity).



The solid is subjected to a radial gravitational body force

$$\rho \mathbf{b} = -\rho B_0 \frac{r-a}{b-a}\mathbf{e}_r$$

6.1 Assume that the velocity field in the shell is radial $\mathbf{v} = v(r)\mathbf{e}_r$.
Calculate the velocity gradient and stretch rate.

$$\mathbf{L} = \nabla \mathbf{v} = \frac{\partial v}{\partial r}\mathbf{e}_r \otimes \mathbf{e}_r + \frac{v}{r}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi\right)$$

$$\mathbf{D} = \mathbf{L}$$

[2 POINTS]

6.2 Show that the incompressibility condition implies that

$$\frac{\partial v}{\partial r} + \frac{2}{r}v = 0$$

and hence find an expression for *v(r)* in terms of $v(a) = \dot{a}$.

$$\int_{v(a)}^{v(r)} \frac{dv}{v} = -\int_a^r \frac{2}{r}dr = \log(v/\dot{a}) = \log(a^2/r^2) \Rightarrow v(r) = \dot{a}\frac{a^2}{r^2}$$

[2 POINTS]

6.3 Hence, find an expression for the particle acceleration $\left.\frac{dv}{dt}\right|_{\mathbf{x}}$, in terms of r, $\dot{a}$ and $\ddot{a}$ Be careful with this – it is not just the partial time derivative of $v(r)$.

$$\left.\frac{dv}{dt}\right|_{\mathbf{x}} = \left.\frac{\partial v}{\partial t}\right|_{r} + \frac{\partial v}{\partial r} v = \ddot{a}\frac{a^2}{r^2} + \frac{2a\dot{a}^2}{r^2} - \frac{2\dot{a}a^2}{r^3}\dot{a}\frac{a^2}{r^2} = \ddot{a}\frac{a^2}{r^2} + \frac{2a\dot{a}^2}{r^2}\left(1 - \frac{a^3}{r^3}\right)$$

[2 POINTS]

6.4 Find an expression for the (total) rate of work done on the shell by gravity.

$$P_B = \int_V \mathbf{b}\cdot\mathbf{v}dV = \int_a^b B_0\rho\frac{r-a}{b-a}4\pi r^2\dot{a}\frac{a^2}{r^2}dr = -2\pi B_0\rho\dot{a}a^2(b-a)$$

[2 POINTS]

6.5 Find an expression for the total kinetic energy of the shell, in terms of $\dot{a}$

$$T = \int_V \frac{1}{2}\rho|\mathbf{v}|^2 dV = \int_a^b \frac{1}{2}\rho\left(\dot{a}\frac{a^2}{r^2}\right)^2 4\pi r^2 dr = 2\pi\rho a^3\dot{a}^2(1 - a/b)$$

[2 POINTS]

6.6 Calculate the time derivative of kinetic energy. Note that b is not constant.

$$\frac{dT}{dt} = 6\pi\rho a^2\dot{a}^3(1-a/b) + 4\pi\rho a^3\ddot{a}\dot{a}(1-a/b) + 2\pi\rho a^4\dot{a}^2\dot{b}/b^2$$

You can eliminate $\dot{b}$ if you like because you know that $\dot{b} = v_r(b) = \dot{a}\frac{a^2}{b^2}$

[2 POINTS]

6.7 Find the total internal stress power, in terms of $\dot{a}$, μ

$$P_\sigma = \int_V \sigma : \mathbf{D}dV = \int_a^b 2\mu\left[\left(\frac{dv}{dr}\right)^2 + \left(\frac{v}{r}\right)^2\right]4\pi r^2 dr = 16\pi\mu a\dot{a}^2(1 - a^3/b^3) \quad \text{(see mupad file)}$$

[2 POINTS]

6.8 Use the principle of virtual work to show that the stress state must satisfy

$$\frac{d\sigma_{rr}}{dr}+\frac{1}{r}\left(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)-\rho B_0\frac{r-b}{a-b}=\rho\left.\frac{dv}{dt}\right|_{\mathbf{x}}$$

$$\int_V \sigma:\delta\mathbf{D}dV-\int_V \mathbf{b}\cdot\delta\mathbf{v}dV+\int_V \rho\frac{d\mathbf{v}}{dt}\delta\mathbf{v}=0$$

$$\Rightarrow\int_a^b\left(\sigma_{rr}\frac{d\delta v}{dr}+\frac{\delta v}{r}(\sigma_{\theta\theta}+\sigma_{\phi\phi})\right)4\pi r^2dr-\int_a^b b_r\delta v4\pi r^2dr+\int_a^b\rho\frac{dv}{dt}\delta v4\pi r^2dr=0$$

Integrate the first term by parts and note the radial stress vanishes at $r=a,r=b$

$$\int_a^b\left(-\frac{d\left(r^2\sigma_{rr}\right)}{dr}\delta v+\frac{\delta v}{r}(\sigma_{\theta\theta}+\sigma_{\phi\phi})r^2\right)4\pi dr-\int_a^b b_r\delta v4\pi r^2dr+\int_a^b\rho\frac{dv}{dt}\delta v4\pi r^2dr=0$$

$$\Rightarrow\int_a^b\left(-\frac{d\left(\sigma_{rr}\right)}{dr}-2\frac{\sigma_{rr}}{r}+\frac{1}{r}(\sigma_{\theta\theta}+\sigma_{\phi\phi})-b_r+\rho\frac{dv}{dt}\right)\delta v4\pi r^2dr=0$$

This must be satisfied for all δv giving the answer stated.

[2 POINTS]

6.9 Write down the boundary conditions for $p(r)$ at $r=a,b$.

$$\sigma_{rr}=0\Rightarrow p+2\mu\frac{\partial v}{\partial r}=p-4\mu\dot{a}\frac{a^2}{r^3}=0$$

$$\Rightarrow p=4\mu\frac{\dot{a}}{a}\qquad r=a$$

$$p=4\mu\frac{\dot{a}}{a}\frac{a^3}{b^3}\qquad r=b$$

[2 POINTS]

6.10 Hence, show that *a(t)* satisfies the differential equation

$$\frac{d^2a}{dt^2}=-B_0\frac{b}{2a}-\frac{da}{dt}\frac{4\mu}{\rho}\frac{a^2+b^2+ab}{a^2b^2}-\left(\frac{da}{dt}\right)^2\frac{(b-a)(a^2+2ab+3b^2)}{2ab^3}$$

See mupad code for solution

[2 POINTS]

6.11 Verify the result of 6.11 using energy methods

The energy equation gives

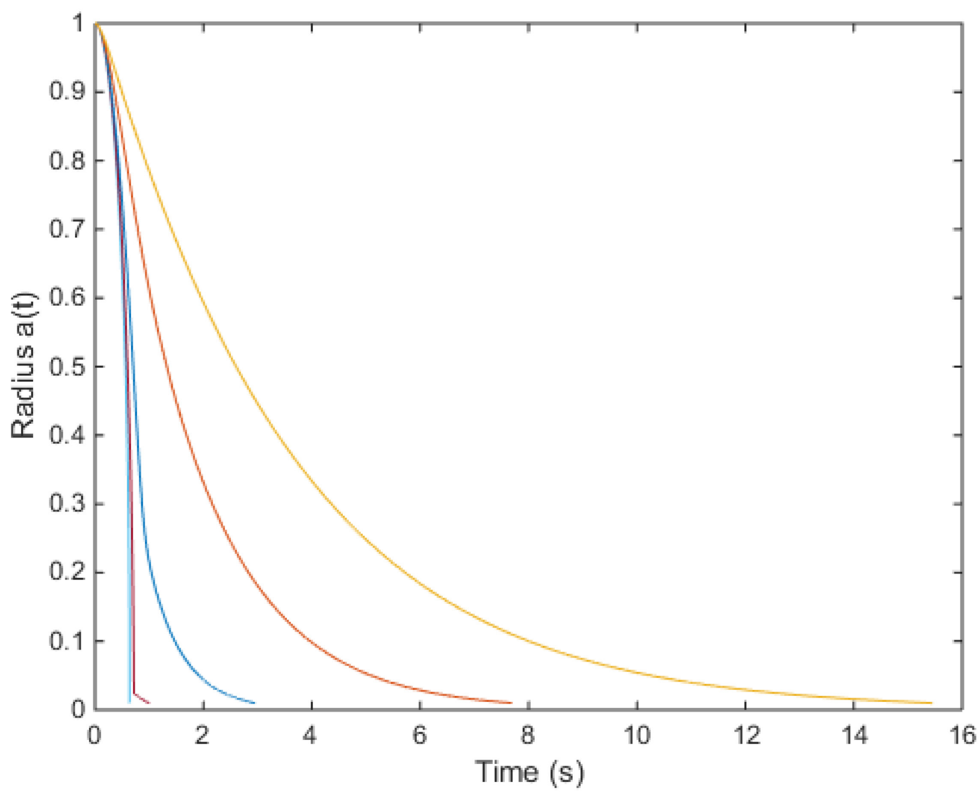
$$\frac{dT}{dt} + P_\sigma = P_B$$

This equation gives the same result (see mupad file for details).

[2 POINTS]

6.12 Plot $a(t)$ with initial conditions $a = 1; b = 5; \dot{a} = 0$ for $B_0 = 1$ and $\mu / \rho = 0.1, 0.2, 0.4, 1.0, 2.0$ (you will need to solve the differential equation numerically, eg using Matlab or Mathematica).

See matlab file for solution



[2 POINTS]