School of Engineering
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EN2210: Continuum Mechanics

Homework 5: Thermodynamics and Constitutive Equations
Solutions

1. Define the *expended power* of external forces acting on a deformable solid (which could be a sub-volume within a larger body) by

$$W_{\exp} = \int_S \mathbf{t}\cdot\mathbf{v} + \int_R \rho\mathbf{b}\cdot\mathbf{v} - \frac{d}{dt}\int_R \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}$$

Show that the expended power is zero for any rigid velocity field of the form

$$\mathbf{v}(\mathbf{y},t) = \mathbf{v}_0(t) + \boldsymbol{\omega}(t)\times(\mathbf{y}-\mathbf{y}_0)$$

where $\mathbf{v}_0(t)$, $\boldsymbol{\omega}(t)$ are vector valued functions of time (but independent of position)

It is helpful to re-write the time derivative of kinetic energy in terms of accelerations

$$\frac{d}{dt}\int_R \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} = \frac{d}{dt}\int_{R_0} \frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v}dV = \int_{R_0}\rho_0\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dV = \int_R \rho\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dV$$

Hence

$$W_{\exp} = \int_S \mathbf{t}\cdot\mathbf{v} + \int_R \rho\left(\mathbf{b}-\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{v}dV$$

It is also convenient to re-write the velocity field as

$$\mathbf{v}(\mathbf{y},t) = \mathbf{v}_0(t) + \mathbf{W}(\mathbf{y}-\mathbf{y}_0)$$

where $\mathbf{W}$ is a skew tensor that has $\boldsymbol{\omega}(t)$ as its dual vector.

We can re-write

$$\begin{aligned} W_{\exp} &= \int_S (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\mathbf{v}dA + \int_R \rho\left(\mathbf{b}-\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{v}dV \\ &= \int_R \nabla_{\mathbf{y}}\cdot(\boldsymbol{\sigma}\mathbf{v})dV + \int_R \rho\left(\mathbf{b}-\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{v}dV \\ &= \int_R \boldsymbol{\sigma}:\nabla_{\mathbf{y}}\mathbf{v}dV + \int_R\left[\nabla_{\mathbf{y}}\cdot\boldsymbol{\sigma} + \rho\left(\mathbf{b}-\frac{d\mathbf{v}}{dt}\right)\right]\cdot\mathbf{v}dV \end{aligned}$$

Finally note that $\nabla_{\mathbf{y}}\mathbf{v}(\mathbf{y},t) = \mathbf{W}$ and the stress is symmetric to satisfy angular momentum. Recall that the contracted product of a symmetric and skew symmetric tensor is zero. The second integral vanishes from balance of linear momentum.

[5 POINTS]

2. It is helpful to have versions of the first and second laws of thermodynamics in terms of quantities defined on the reference configuration, including (but not limited to!) :

- Reference mass density ρ_0
- Nominal stress and deformation gradient $\mathbf{S}, \dot{\mathbf{F}}$
- Material stress and Lagrange strain rate $\mathbf{\Sigma}, \dot{\mathbf{E}}$
- Referential heat flux $\mathbf{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$

Give two equations for the first law, and give two expressions for the Clausius Duhem inequality.

We need to repeat the steps used to derive the first and second laws in the current configuration into the reference configuration.

For the first law

$$W = \int_V b_i v_i dV + \int_A \sigma_{ij} n_i v_j dA = \int_V \sigma_{ij} D_{ij} dV + \frac{d}{dt}\left\{\int_V \frac{1}{2}\rho v_i v_i dV\right\} = \int_{V_0} S_{ij}\frac{dF_{ji}}{dt}dV_0 + \frac{d}{dt}(KE)$$

$$Q = \int_V q dV - \int_A \mathbf{q}\cdot\mathbf{n}dA = \int_{V_0} qJdV_0 - \int_{A_0} \mathbf{Q}\cdot\mathbf{n}_0 dA_0 = \int_{V_0}\left(Jq - \frac{\partial Q_i}{\partial x_i}\right)dV$$

Hence

$$\frac{d}{dt}\left(\int_{V_0} \rho_0 \varepsilon dV_0 + KE\right) = \int_{V_0} S_{ij}\frac{dF_{ji}}{dt}dV_0 + \frac{d}{dt}(KE) + \int_{V_0}\left(Jq - \frac{\partial Q_i}{\partial x_i}\right)dV_0$$

$$\rho_0 \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = S_{ij}\frac{dF_{ji}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

Similarly in terms of material stress

$$\rho_0 \left.\frac{\partial \varepsilon}{\partial t}\right|_{\mathbf{x}=const} = \Sigma_{ij}\frac{dE_{ij}}{dt} - \frac{\partial Q_i}{\partial x_i} + Jq$$

The second law can be expressed as

$$S_{ij}\frac{dF_{ji}}{dt} - \frac{1}{\theta}Q_i\frac{\partial \theta}{\partial x_i} - \rho_0\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \ge 0$$

$$S_{ij}\frac{dE_{ij}}{dt} - \frac{1}{\theta}Q_i\frac{\partial \theta}{\partial x_i} - \rho_0\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right)$$

[5 POINTS]

3. Starting with the local form of the second law of thermodynamics and mass conservation

$$\rho \left.\frac{\partial s}{\partial t}\right|_{\mathbf{x}=const} + \frac{\partial (q_i/\theta)}{\partial y_i} - \frac{q}{\theta} \geq 0 \qquad \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

(the symbols have their usual meaning), derive the statement of the second law for a control volume

$$\frac{\partial}{\partial t}\int_R \rho s dV + \int_B \rho s(\mathbf{v}\cdot\mathbf{n})dA + \int_B \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA - \int_R \frac{q}{\theta}dV \geq 0$$

Note that we can write the first term as

$$\rho \left.\frac{\partial s}{\partial t}\right|_{\mathbf{x}} = \rho \left.\frac{\partial s}{\partial t}\right|_{\mathbf{y}} + \rho \frac{\partial s}{\partial y_i} v_i = \left.\frac{\partial \rho s}{\partial t}\right|_{\mathbf{y}} - s\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \rho \frac{\partial s}{\partial y_i}$$

Then, using mass conservation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho v_i}{\partial y_i} = 0 \Rightarrow \left.\frac{\partial \rho s}{\partial t}\right|_{\mathbf{y}} - s\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}} + \rho \frac{\partial s}{\partial y_i} = \left.\frac{\partial \rho s}{\partial t}\right|_{\mathbf{y}} + s\frac{\partial \rho v_i}{\partial y_i} + \rho \frac{\partial s}{\partial y_i} = \left.\frac{\partial \rho s}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho s v_i}{\partial y_i}$$

Hence

$$\left.\frac{\partial \rho s}{\partial t}\right|_{\mathbf{y}} + \frac{\partial \rho s v_i}{\partial y_i} + \frac{\partial (q_i/\theta)}{\partial y_i} - \frac{q}{\theta} \geq 0$$

Integrating this over a fixed spatial volume, noting that the time derivative can be taken outside the integral, and applying the divergence theorem gives the required solution.

[5 POINTS]

4. State how the following quantities transform under a change of observer

(i) The spatial heat flux vector $\mathbf{q}$

(ii) The referential heat flux vector $\mathbf{\Theta} = J\mathbf{F}^{-1}\mathbf{q}$

(iv) The infinitesimal strain tensor $\boldsymbol{\varepsilon} = \left[\nabla_{\mathbf{x}}\mathbf{u} + \left(\nabla_{\mathbf{x}}\mathbf{u}\right)^T\right]/2$

(iv) The spatial gradient of a scalar function of position in a deformed solid $\mathbf{g} = \nabla_{\mathbf{y}}\phi(\mathbf{y})$

(v) The material gradient of a function of particle position in a solid $\mathbf{G} = \nabla\phi(\mathbf{x})$

(i) $\mathbf{q}$ is a spatial vector therefore $\mathbf{q}^* = \mathbf{Q}\mathbf{q}$

(ii) $\mathbf{\Theta} = J\mathbf{F}^{-1}\mathbf{q} \Rightarrow \mathbf{\Theta}^* = J^*\mathbf{F}^{*-1}\mathbf{q}^* = J\mathbf{F}^{-1}\mathbf{Q}^T\mathbf{Q}\mathbf{q} = \mathbf{\Theta}$

(iii) The displacement transforms as $\mathbf{u}^* = \mathbf{Q}\mathbf{u}$ and the referential derivative is invariant so

$$\boldsymbol{\varepsilon}^* = \left[\nabla_{\mathbf{x}}\mathbf{u}^* + \left(\nabla_{\mathbf{x}}\mathbf{u}^*\right)^T\right]/2 = \left[\mathbf{Q}\nabla_{\mathbf{x}}\mathbf{u} + \left(\nabla_{\mathbf{x}}\mathbf{Q}\mathbf{u}\right)^T\right]/2 = \left[\mathbf{Q}\nabla_{\mathbf{x}}\mathbf{u} + \left(\nabla_{\mathbf{x}}\mathbf{u}\right)^T\mathbf{Q}^T\right]/2$$

(iv) $\mathbf{g}^* = \nabla_{\mathbf{y}^*}\phi = \nabla_{\mathbf{y}}\phi\mathbf{Q}^T = \mathbf{Q}\nabla_{\mathbf{y}}\phi = \mathbf{Q}\mathbf{g}$

(v) The referential gradient is invariant by inspection, or else note $\mathbf{G} = \mathbf{g}\mathbf{F} \Rightarrow \mathbf{G}^* = \mathbf{g}^*\mathbf{F}^* = \mathbf{g}\mathbf{Q}^T\mathbf{Q}\mathbf{F} = \mathbf{G}$

[4 POINTS]