School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 6: Constitutive Equations, Fluid Mechanics
Due November 9, 2016

1. Show that a fluid with constitutive equation of the form

$$\sigma_{ij} = -\pi_{eq}(\rho,\theta)\delta_{ij} + \tau_{ij}^0 + 2\mu D_{ij}$$

with τ_{ij}^0 a nonzero constant, violates the second law of thermodynamics.

2. Suppose that the internal energy of a continuum is expressed as a function of density and entropy, as $\varepsilon(\rho,s)$. Show that the dissipation inequality requires that

$$\sigma_{ij}D_{ij} - \frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i} - \rho\left(\frac{\partial\varepsilon}{\partial t} - \theta\frac{\partial s}{\partial t}\right) \ge 0$$

3. Consider an inviscid van der Waals fluid with specific heat capacity $c_v(\theta)$ an arbitrary function of temperature (but independent of density), and pressure related to temperature and density by

$$\hat{\pi}_{eq} = \frac{\rho R\theta}{1-b\rho} - a\rho^2$$

(here a is just a constant)

3.1 Show that the dissipation inequality (use problem 2, and the approach used in class to obtain constitutive equations for a fluid) requires that

$$\hat{\pi}_{eq} = \rho^2\frac{\partial\varepsilon}{\partial\rho} \qquad \theta = \frac{\partial\varepsilon}{\partial s}$$

$$\pi_{eq}\frac{\dot{\rho}}{\rho} - \rho\frac{\partial\varepsilon}{\partial\rho}\dot{\rho} - \rho\frac{\partial\varepsilon}{\partial s}\dot{s} - \theta\rho\frac{\partial s}{\partial t} = 0$$

and hence

$$\frac{d\varepsilon}{dt} = \hat{\pi}_{eq}\frac{1}{\rho^2}\frac{d\rho}{dt} + \theta\frac{ds}{dt}$$

3.2 Hence conclude that

$$\frac{d\left(\varepsilon + a\rho\right)}{dt} = \theta\frac{d}{dt}\left(R\log\frac{\rho}{1-b\rho} + s\right)$$

for all $\dot{\rho},\dot{\theta}$

3.3 Hence show that

$$s = \int \frac{c_v(\theta)}{\theta} d\theta - R\log\frac{\rho}{1-b\rho} + const$$
$$\varepsilon = \int c_v(\theta) d\theta - a\rho + const$$

3. The deformation of a viscoelastic material is modeled by representing the deformation gradient **F** of a material element as a sequence of an irreversible deformation $\mathbf{F}^p$, followed by a reversible (elastic) deformation $\mathbf{F}^e$, so that $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$. The Helmholtz free energy $\psi(\mathbf{F}^e, \theta)$ of the material is assumed to be a function of $\mathbf{F}^e$ and temperature θ only.

3.1 Show that the velocity gradient **L** can be decomposed into elastic and plastic parts as

$$\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p \qquad \mathbf{L}^e = \frac{d\mathbf{F}^e}{dt}\mathbf{F}^{e-1} \qquad \mathbf{L}^p = \mathbf{F}^e \frac{d\mathbf{F}^p}{dt}\mathbf{F}^{p-1}\mathbf{F}^{e-1}$$

3.2 Show that the dissipation inequality

$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial \theta}{\partial y_i} - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) \geq 0$$

requires that the Cauchy stress is related to the free energy by

$$J F_{kj}^{e-1} \sigma_{ji} = \rho_0 \frac{\partial \psi}{\partial F_{ik}^e}$$

(where ρ_0 is the mass per unit reference volume) and that the plastic part of the velocity gradient must satisfy

$$\sigma_{ij} L_{ij}^p \geq 0$$

3.3 Assume that $\mathbf{F}^e$ and $\mathbf{F}^p$ transform under a change of observer according to $\mathbf{F}^{e*} = \mathbf{Q}\mathbf{F}^e \quad \mathbf{F}^{p*} = \mathbf{F}^p$. Verify that the transformation is consistent with the transformation of deformation gradient **F** under an observer change, and determine expressions for $\mathbf{L}^{e*}, \mathbf{L}^{p*}$ in terms of $\mathbf{Q}$ and $\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$.

3.4 Consider a constitutive relation in which the plastic velocity gradient is given by

$$L_{ij}^p = \eta\left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)$$

Show that if $\det(\mathbf{F}^p) = 1$ at time *t=0*, then $\det(\mathbf{F}^p) = 1$ for all *t>0*. (Hint: consider L_{kk}^p)

3.5 Show that the constitutive relation in 3.4 satisfies both frame indifference and the dissipation inequality (assume $\eta > 0$).